THE
THEORY OF PLANE CURVES

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In

Memory of

SIR ASUTOSH MOOKERJEE

PREFACE TO THE SECOND EDITION

The first edition of the present volume, published some years back under the title "Lectures on the Theory of Plane Curves," was designed to meet the syllabus prescribed by the University of Calcutta for the Master's Degree, and intended as an introductory course suitable for students of higher geometry, scarcely assuming any further knowledge of higher analysis on the part of the reader than is to be found in most of the ordinary text books on Calculus and Plane Analytical Geometry. Since then it has been suggested that the book should be so revised and enlarged as to include materials which would not only be of use to the students for the Master's course, but also encourage independent thinking in students of higher studies engaged in research work. In the preparation of the second edition, therefore, special care has been taken to incorporate recent researches as far as possible and to indicate references to original sources as far as practicable. In fact, almost all the chapters have been re-written and the articles re-numbered, while five additional sections—Chapters VII, X, XI, XII, XIII—have been inserted and a large number of examples given illustrating the subject-matter and serving as exercises for students. The volume contains an exposition of the general theory of plane algebraic curves in its various aspects with applications to conics, cubics, quartics, etc.

In writing on Higher Geometry, it is always a problem to determine what matters to exclude, and in dealing with a subject so wide in its scope, which attracted so many workers and has been so much developed in recent years, specially by the Italian

Geometers, it has not been possible to do full justice to all the important topics ; in consequence, some have received fuller attention, while others of equal or greater importance have been little noticed or even omitted altogether. It is hoped, however, that the book will afford some scope for independent thinking and research, when the student enters upon a systematic study of plane curves.

In the preparation of the volume, constant recourse has been had to the classical works of Salmon, Clebsch, Cayley, Cremona, the works of Basset, Teixeira, Scott, Wieleitner, Loria and the papers of Zeuthen, Brill and Nöther, Castelnuovo and others, published in the various Journals and Periodicals. Prof. Pascal's *Repertorium der höheren Geometrie* was of great use in supplying a number of important references. Since the publication of the first edition, Prof. Hilton has published his "Plane Algebraic Curves," which has also been studied with much advantage. My obligations to these authors, greater than I can confess, are gratefully acknowledged, and it is impossible to record in detail my obligation to the great inspiring work of Salmon—Higher Plane Curves. I had no access to the recent work of Enriques—*Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, 2 Vols.*—so highly spoken of, and it is likely that the present volume could have been made much more suggestive, had I had the opportunity of consulting this book.

In concluding this preface, I must take the opportunity of recording my indebtedness to Prof. J. L. Coolidge of Harvard for his very valuable suggestions for the improvement of the work, and to Mr. A. C. Bose, Controller of Examinations in the University of Calcutta, for his extreme kindness in giving valuable hints and suggestions.

My best thanks are also due to my former pupil Mr. L. Murthi, M.A., who made a number of important

suggestions and pointed out several printing and other errors in the preparation of the edition.

With a keen sense of sorrow and gratitude, I record my indebtedness to the late lamented Sir Asutosh Mookerjee, former Vice-Chancellor and President of the Post-graduate Councils in the University of Calcutta, whose untimely death has been deeply felt by all Indian workers in the field of Mathematics, pure or applied. It was at his suggestion that I was induced, difficult as the task was, to revise the original lecture notes for the Press, and it was he who helped and encouraged me to bring out a second and revised edition, but it is a matter of profound regret that he was not spared to see its completion. As an humble token of gratitude, this volume is dedicated to his revered memory.

UNIVERSITY OF CALCUTTA,

March, 1925.

S. M. G.

PREFACE TO THE THIRD EDITION

The chief object of reprinting the preface to the second edition is to recall the expression of my feelings of sorrow when I was deprived of the inspiring encouragement of the great Sir Asutosh Mookerjee in the preparation of that edition.

The present edition is practically a reprint from its former edition, with slight modifications, mostly verbal, which seemed desirable in order to make the work more useful to those for whom it is intended. I take this opportunity of recording my grateful appreciation of a number of suggestions by Prof. T. R. Holcroft and others, although, I confess, it was not possible, within the limited scope of the work, to incorporate all of them, but appropriate references are given for profitable use by more inquisitive and advanced scholars.

In concluding this preface, I must convey my best thanks to the Authorities of the Calcutta University Press for the extreme care and promptness with which they finished printing of the work, and specially to Mr. Kalipada Das who rendered valuable assistance in reading the final proofs.

UNIVERSITY OF CALCUTTA,

S. M. G.

October, 1931.

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CHAPTER I

INTRODUCTION

1. Co-ordinates:

Homogeneous co-ordinates are most conveniently used in studying projective properties of plane curves. An intimate knowledge of the use of these co-ordinates, however, will be assumed on the part of the learner, though some of the most important results are quoted for ready reference.

There are two kinds of homogeneous co-ordinates most commonly in use: (1) *Trilinear Co-ordinates*; (2) *Areal Co-ordinates*. The trilinear co-ordinates of a point are generally denoted by α, β, γ and its areal co-ordinates by x, y, z. The formulæ * of transformation for the two systems are—

$$\frac{a\alpha}{2\triangle}=x, \quad \frac{b\beta}{2\triangle}=y, \quad \frac{c\gamma}{2\triangle}=z \qquad \ldots \quad (1)$$

where $\triangle$ is the area of the triangle of reference and a, b, c its sides. The identical relation satisfied by the trilinear co-ordinates of a point is—

$$a\alpha+b\beta+c\gamma=2\triangle \qquad \ldots \quad (2)$$

* Scott, Modern Analytical Geometry, § 16.

and consequently, the same relation in the *Areal* system becomes—

$$x+y+z=1 \qquad \dots \quad (3)$$

2. The Special Line at Infinity:

The equation of the line at infinity in the two systems respectively is

$$a\alpha+b\beta+c\gamma=0 \qquad \dots \quad (4)$$

and

$$x+y+z=0 \qquad \dots \quad (5)$$

It is to be noticed that these equations contradict the fundamental identical relations (2) and (3). This paradox can be explained by the fact that the relations (2) and (3) are deduced on the supposition that α, β, γ (or x, y, z) are all finite; but the relations do not hold when any of the variables happens to be infinite. The line represented by (4) or (5) is entirely at infinity, and every point on this line is at infinity. For, to determine the trilinear co-ordinates of a point on the line, we have to solve the two equations—

$$\begin{cases} a\alpha+b\beta+c\gamma=2\triangle \\ a\alpha+b\beta+c\gamma=0. \end{cases}$$

i.e., to determine the values of α, β, γ which satisfy—

$$0.\alpha+0.\beta+0.\gamma=2\triangle \qquad \dots \quad (6)$$

Equation (6) requires that one at least of the quantities α, β, γ should be infinite, and consequently from the first of the two equations, one other becomes infinite. Thus, it is seen that two of the co-ordinates of a point on the line are infinite, and therefore the point is at infinity. Conversely, every point at infinity lies on this line.

From these considerations it follows that the co-ordinates of a point in a plane are in general connected by

a linear relation $a\alpha+b\beta+c\gamma=constant$, which, in fact, reduces to the inequality $a\alpha+b\beta+c\gamma\neq 0$; but there are special points in the plane, whose co-ordinates satisfy the relation $a\alpha+b\beta+c\gamma=0$. These exceptional points all lie at an *infinite* distance on a certain special line lying entirely at infinity.

3. Cartesian as a special System of Homogeneous Co-ordinates:

If the two Cartesian axes and the line at infinity are taken as the sides of the fundamental triangle, the formulæ * of transformation from the homogeneous to the Cartesian system may be written as:—

$$\alpha = \text{any multiple of } x, \quad \beta = \text{any multiple of } y,$$

$$\gamma = \text{any convenient constant.}$$

Thus, if in any equation in the homogeneous system, we substitute $\alpha=x$, $\beta=y$, $\gamma=1$, the equation is transformed into one in the Cartesian system. Conversely, in passing from the Cartesian to the homogeneous system, such powers of z are introduced in the different terms as will make the equation homogeneous. The line at infinity in this case is

$$z=0, \quad i.e., \quad 0.\,x+0.\,y+z=0.$$

Note.—In analytical investigations, a certain equation is made homogeneous by introducing proper powers of z in the different terms, and finally for expressing the result in the original system, z is put equal to unity.

* Scott, *loc. cit.*, § 30.

4. Tangential * or Line Co-ordinates:

Just as geometrical figures are regarded as loci of points, they may also be regarded as generated by the motion of a straight line, instead of points. In this case we may suppose that the straight line is the primary element, the point is to be considered as secondary, being obtained as the intersection of lines. The two conceptions may be simultaneously developed. An infinite number of points are supposed to lie on a straight line, and an infinite number of lines may be made to pass through a point. Thus it appears that there is a correspondence between the two theories,—the *point-theory*, in which points are taken as primary elements, and the *line-theory*, in which lines are taken as primary elements, and this we shall indicate to some extent presently.

In the line system, the position of a line is determined in reference to three fixed fundamental points. The ratios of the distances of the three fixed points from the line, measured in a fixed direction, are sufficient to determine the position of the line *uniquely*. Thus the co-ordinates of a line may be defined as proportional to given multiples of the distances, measured in given directions, of the three fundamental points from the line. For simplicity, these distances are measured in a direction *perpendicular* to the line. Thus if p, q, r be the lengths of the perpendiculars drawn from the three fundamental points on to the line, then the co-ordinates (ξ, η, ζ) of the line are proportional to $p:q:r$;

$$i.e. \quad \xi : \eta : \zeta = p : q : r.$$

5. Relation between the Co-ordinates of a Line and those of a Point on it:

Let $\quad (x_1, y_1, z_1),\ (x_2, y_2, z_2),\ (x_3, y_3, z_3)$

* For a fuller treatment of the subject, the reader in referred to Dr. Booth's "A Treatise on some New Geometrical Methods," Vol. I.

be the co-ordinates of the three fundamental points and let the equation of a line be

$$lx+my+nz=0.$$

Then the lengths of the perpendiculars drawn from the three points on to the line are respectively proportional * to

$$lx_1+my_1+nz_1,\ \ lx_2+my_2+nz_2 \ \text{ and } \ lx_3+my_3+nz_3.$$

Therefore, the co-ordinates of the line are given by—

$$\xi:\eta:\zeta=lx_1+my_1+nz_1:\ lx_2+my_2+nz_2:lx_3+my_3+nz_3.$$

$$\textit{i.e.}\qquad k\xi=lx_1+my_1+nz_1,\ k\eta=lx_2+my_2+nz_2,\ k\zeta=lx_3+my_3+nz_3.$$

If we eliminate l, m, n, k between these equations and the equation of the line, we obtain the equation of the line in the form—

$$\begin{vmatrix} 0 & x & y & z \\ \xi & x_1 & y_1 & z_1 \\ \eta & x_2 & y_2 & z_2 \\ \zeta & x_3 & y_3 & z_3 \end{vmatrix}=0.$$

In this equation the co-efficients of ξ, η, ζ are linear functions of x, y, z. If these co-efficients are denoted by α, β, γ, the equation may be written as—

$$\xi\alpha+\eta\beta+\zeta\gamma=0 \qquad \dots \quad (1)$$

Now, the linear functions α, β, γ determine a point whose co-ordinates are proportional to α, β, γ. Therefore,

* Salmon's Conic Sections, § 61.

all lines whose co-ordinates ξ, η, ζ are connected by the relation (1) pass through a point whose co-ordinates are α, β, γ; and conversely, all points whose co-ordinates satisfy the relation (1) lie on a line whose co-ordinates are ξ, η, ζ. The same fact is expressed by saying that *the point* (α, β, γ) *and the line* (ξ, η, ζ) *are united in position,* if the relation (1) is satisfied.

6. Tangential Equation :

The idea that a curve may be regarded as the envelope of a moving line is due to De Beaune (1601-1652). A systematic treatment of envelopes was given in 1692 by Leibnitz. The advantage of developing the two conceptions side by side was first pointed out by Brianchon (1806). It was Möbius who in his Barycentric Calculus introduced a system of line co-ordinates. Applications of these co-ordinates to metrical properties were given by Chasles and Salmon.

The point (α, β, γ) may be regarded as the envelope of all lines whose co-ordinates satisfy the above relation and the equation $\xi\alpha + \eta\beta + \zeta\gamma = 0$ represents in *line co-ordinates* a point whose trilinear co-ordinates are α, β, γ; and in general, any homogeneous equation in line co-ordinates ξ, η, ζ represents the envelope of all lines whose point equation is $\xi\alpha + \eta\beta + \zeta\gamma = 0$ and whose co-ordinates satisfy the given equation. This relation is called the "tangential equation" of the envelope. Thus the tangential equation of a curve is the relation between ξ, η, ζ, which expresses the condition that the line $\xi\alpha + \eta\beta + \zeta\gamma = 0$ touches the curve; * *i.e.*, $f(x, y, z) = 0$ being the point equation of a curve, the condition that the line $\xi x + \eta y + \zeta z = 0$ touches it gives a relation of the form $\phi(\xi, \eta, \zeta) = 0$, which is called the *line or tangential* equation of the curve.

* Salmon, *loc. cit.*, §§ 151 and 285.

7. The Circular Points at Infinity :

Just as the co-ordinates of a point satisfy an identical relation, the co-ordinates of a line also satisfy an identical relation. In fact, the co-ordinates of a line satisfy the relation—

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C=4\triangle^2,$$

where $\triangle$ is the area of the fundamental triangle, whose angles are denoted by A, B, C.

Let p, q, r be the perpendiculars on to the line (ξ, η, ζ) drawn from the vertices of the fundamental triangle. Now the perpendicular (δ) drawn from any point (f, g, h) on to the line (ξ, η, ζ) is given by—

$$\delta^* = (\xi f+\eta g+\zeta h)/k,$$

where

$$k=\sqrt{\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C}.$$

$\therefore$ The perpendicular drawn from the vertex $A(2\triangle/a,0,0)$ on the line ξ, η, ζ is given by—

$$p=2\triangle\xi/ak; \quad i.e., \quad apk=2\triangle\xi.$$

Similarly,

$$bqk=2\triangle\eta \quad \text{and} \quad crk=2\triangle\zeta.$$

Hence we obtain

$$a^2p^2+b^2q^2+c^2r^2-2bcqr\cos A-2carp\cos B$$
$$-2abpq\cos C=4\triangle^2 \qquad \dots \quad (1)$$

which gives a relation between p, q, r. If (ξ, η, ζ) be the actual co-ordinates of the line, we have

$$\xi : \eta : \zeta = ap : bq : cr.$$

* Ferrer's Trilinear Co-ordinates, p. 20.

$\therefore$ The relation (1) becomes—

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C=4\triangle^2 \quad (2)$$

i.e., the co-ordinates (ξ, η, ζ) of the line satisfy the inequality

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C\neq 0.$$

But certainly there exists at least one line whose co-ordinates are not subject to this inequality, *namely*, the line at infinity

$$a\alpha+b\beta+c\gamma=0, \text{ for which } \xi : \eta : \zeta = a : b : c.$$

Let us consider the lines whose co-ordinates satisfy the condition—

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C^*=0 \quad (3)$$

This is an equation of the second degree and consequently all these lines constitute a system of the second class. The expression on the left breaks up into two linear factors and the envelope is therefore a pair of points.† In fact, the equation reduces to—

$$(\xi e^{iB}+\eta e^{-iA}-\zeta)\,(\xi e^{-iB}+\eta e^{iA}-\zeta)=0,$$

and consequently, the envelope is a pair of points whose equations are—

$$\xi e^{iB}+\eta e^{-iA}-\zeta=0 \quad \text{and} \quad \xi e^{-iB}+\eta e^{iA}-\zeta=0 \quad \ldots \quad (4)$$

The special lines under consideration pass through one or other of these two fixed points. The factors being

* For a geometrical interpretation of this equation, the reader is referred to Professor Cayley's *Sixth Memoir upon Quantics* (Coll. Papers, Vol. II, No. 158). This is a degenerate envelope and the name "the Absolute" has been given to it by Prof. Cayley.

† Salmon, *loc. cit.*, § 286.

imaginary, the points they represent are conjugate imaginary points. The line joining them is real, whose co-ordinates are obtained by solving the equations (4).

Thus,

$$\frac{\xi}{e^{iA}-e^{-iA}}=\frac{\eta}{e^{iB}-e^{iB}}=\frac{\zeta}{e^{i(A+B)}-e^{-i(A+B)}}$$

or, $\dfrac{\xi}{\sin A}=\dfrac{\eta}{\sin B}=\dfrac{\zeta}{\sin C}$ *i.e.*, $\xi:\eta:\zeta=a:b:c$.

The line is therefore the line at infinity $a\alpha+b\beta+c\gamma=0$. The two imaginary points (usually denoted by I and J) are at infinity. All lines drawn through them are imaginary. Through any real point P there pass two of these lines PI and PJ. They are called "*isotropic*" lines or *circular lines* and their Cartesian equations are

$$x\pm y\sqrt{-1}+c=0$$

The co-ordinates of these two points are—

$$(e^{iB},\ e^{-iA},\ -1) \quad \text{and} \quad (e^{-iB},\ e^{iA},\ -1) \qquad \dots \quad (5)$$

The co-ordinates may also be taken as—

$$(e^{iC},-1,\ e^{iA}),\quad (e^{-iC},-1,\ e^{iA})$$

or, $(-1,\ e^{iC},\ e^{-iB}),\quad (-1,\ e^{-iC},\ e^{iB})$.

It is proved in treatises on conic sections that all circles pass through the two circular points at infinity, and it is on this account that they have been so called. These points are then found as the intersections of the line at infinity with any circle—the circumcircle of the fundamental triangle, for example, *i.e.*, they are given by the equations:—

$$\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C=0$$

$$\alpha\sin A+\beta\sin B+\gamma\sin C=0$$

Solving these two equations for α, β, γ, we obtain the co-ordinates of the two circular points I and J.

When the fundamental triangle is equilateral, $a=b=c$ and $A=B=C=60°$, and the co-ordinates of I and J become $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$, where ω is one of the imaginary cube roots of unity.

8. Isotropic Lines:

In Cartesian system the line-equation of the circular points at infinity is $\xi^2+\eta^2=0$. For, the isotropic lines through any point are $x \pm y\sqrt{-1}+c=0$; and consequently, any line $\xi x+\eta y+\zeta z=0$ will pass through *one* of these points, if $\xi^2+\eta^2=0$, which is the tangential or line-equation required.

This equation implies that every line drawn through one of these points is perpendicular to itself, for this is the condition that the line $\xi x+\eta y+\zeta z=0$ is perpendicular to itself. The same equation further implies that the length of the perpendicular drawn from any point on any of the circular lines is always infinite.* The equivalent condition in trilinear co-ordinates is accordingly obtained by equating to nothing the denominator in the expression for the length of a perpendicular (§ 7).

9. Properties of the Circular Points † at Infinity:

The two special points at infinity play an important part in the theory of curves.

(1) The two points in which any two perpendicular lines meet the line at infinity form a harmonic range with the circular points I and J. This practically amounts to saying that two perpendicular lines form with the isotropic lines through their intersection a harmonic pencil.‡ We are thus led to the following definition:—*Lines harmonic*

* Salmon, Conics, § 34

† These were discovered by Poncelet, *Traitè des propriètés projective des figures* (Paris, 1822). Prof. Cayley has discussed at some length their properties in the "Sixth Memoir upon Quantics."

‡ Salmon, Conics, § 356.

with respect to the isotropic lines through their intersection are said to be perpendicular.

(2) Curves may be differentiated in respect of their relations to the two circular points. These being conjugate imaginary points, a real curve which passes through the one passes through the other also. Curves passing through these points possess special properties and are called *circular* curves.

(3) The points I and J have important functions in determining the foci of a curve, which we shall have occasion to discuss in a subsequent chapter.

10. The Line at Infinity :

The notion of elements lying at an infinite distance is due to Desargues, who considered that parallel straight lines meet at an infinitely distant point and parallel planes pass through the same straight line at an infinite distance. The same idea was developed by Poncelet, who discovered the two circular points at infinity. The points which are supposed to constitute the line at infinity are not of the same nature as those in the finite part of the plane. The exact nature of this line cannot be ascertained. It is not a line in the ordinary sense and is, in fact, fictitious, invented simply to secure the generality of the statement that any two straight lines in a plane always intersect. It is a line in the sense that it corresponds to a finite line in homographic transformations. It meets every other line in one and only one point. It lies at infinity only in the sense that it contains no finite point. The statement that it lies at infinity indicates its character and not its position. It has no direction and cannot be graphically represented.

11. Theory of Projection :

The principles of projection are well explained in treatises on conic sections.* The modern theory of projective geometry is only a development of the principles enunciated by Desargues who was the first to make use of conical projection. Poncelet, by his wonderful discovery of the circular points at infinity built up a logical system of geometry of conics; but it was Plücker who extended this conception and defined the foci of curves by their isotropic properties.

Let p and p' be two fixed planes and O a fixed point outside both of them. If A is a point in p and OA meets p' in A$'$, then A$'$ is called the *projection* of A on p', O is called the *vertex* of projection.

If A traces out a locus C in the plane p, and O be joined to all points on C by means of straight lines, the joining lines will generate a cone, and the *plane* p' intersects this cone in a curve C$'$, which is called the *projection* of C on p', and p' is called the plane of projection. The line of intersection of p and p' is called the *axis* of projection.

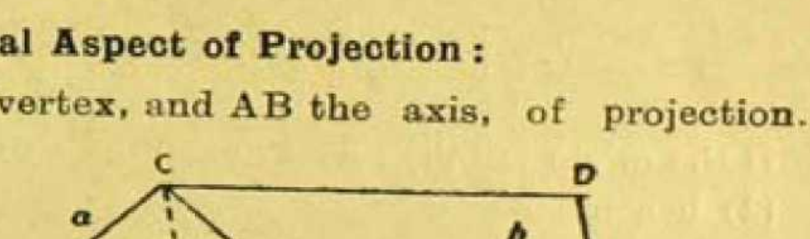
Thus the projection of a right line is another right line. If a right line intersects a curve in n points, then its projection will cut the projection of the curve in n corresponding points. Hence the projection of a curve of the nth degree is another curve of the same degree. A tangent to a curve projects into a tangent to the projection of the curve, and every singularity on the curve projects into the same singularity on the projected curve. The projection of a range of four points (or lines) is a range of four points (or lines), having the same cross-ratio. Pole and polar relations and conjugate properties of lines and points remain unaltered by projection.

* Salmon's Conics, Chap. XVII.

If through the vertex O a plane is drawn parallel to p, cutting the plane p' in a line s', then s' is the projection on p' of the line at infinity of p. Similarly, if the plane, drawn parallel to p', intersects p in a line s, s is the projection of the line at infinity of p'. s and s' are called *vanishing lines* of the planes p and p' respectively. Thus we see that any line in p can be projected into the line at infinity on p', while the line at infinity on p' can be projected into any line in p. It follows therefore that the properties of curves having singularities in the finite part of the plane can be deduced from those of curves having the corresponding singularities at infinity, and *vice versa*. Thus, all properties of a curve, which do not involve magnitudes of lines or angles, can be generalised by projection, while metrical properties cannot be generalised except in very special cases. Any two points in a plane can be projected into the two circular points, and *vice versa*; but this can be effected by an imaginary projection. We postpone further discussion of the theory and its application, which we shall have occasion to illustrate as we proceed.

12. Analytical Aspect of Projection:

Let O be the vertex, and AB the axis, of projection. The plane through the vertex O parallel to p' meets p in the vanishing line CD. Let the plane through O per-



pendicular to the axis AB meet AB and CD in A and C respectively. Take any point V as origin on AB. In the plane p, take VB and any perpendicular line as axes of x

and y ; and in the plane p' take VB and any perpendicular line as axes of x' and y'.

Let (x, y) and (x', y') be the co-ordinates of P and P' referred to these axes respectively. Let CP meet AB in M'.

Then OC is parallel to P'M'. But, OC is perpendicular to AB, and hence P'M' is perpendicular to AB. Draw PM perpendicular to AB, and let $OC=a$, $AC=b$ and $AV=c$.

Then,

$$\frac{P'M'}{OC}=\frac{PM'}{PC}=\frac{MM'}{MA}$$

$$\therefore \quad \frac{y'}{a}=\frac{x'-x}{x+c}, \quad \text{whence,} \quad x=\frac{ax'-cy'}{y'+a} \quad \dots \quad (1)$$

Similarly,

$$\frac{PM}{AC}=\frac{M'P}{M'C}=\frac{M'P'}{M'P'+OC}$$

$$\therefore \quad \frac{y}{b}=\frac{y'}{y'+a}, \quad \text{or} \quad y=\frac{by'}{y'+a} \quad \dots \quad (2)$$

If the origin is taken at A, $c=0$, and the formulæ (1) and (2) become—

$$x=\frac{ax'}{y'+a}, \quad y=\frac{by'}{y'+a} \ .$$

These formulæ are useful in representing the process of projection by analytical transformations. The constants a, b, c define the position of the vertex of projection, and consequently, when a, b, c are all real, the vertex is real and the projection is also real ; but if any of these quantities become imaginary, the vertex is an

imaginary point, and projection cannot be effected geometrically, but still the analytical process is perfectly valid.

Ex. 1. Prove that any conic can be projected into a circle and at the same time any given line to infinity.

Ex. 2. Prove that any conic touching the vanishing line projects into a parabola.

Ex. 3. A system of conics having double contact with each other can be projected into a system of concentric circles.

Ex. 4. The conics $2x^2+3y^2=1$ and $x^2=2y$ are projected into circles. Find the necessary equations of transformation.

Ex. 5. Find a transformation by which the conics $y^2=4x-3$ and $x^2-y^2-4x+3=0$ may be projected into circles.

13. Figures in perspective:

A figure is said to be in *perspective* with another figure when the lines joining the corresponding points of the two figures pass through a common point O. This point is called the *centre* of perspective.

Thus a figure when projected on to a plane or surface is in perspective with its projection, the vertex of projection being the centre of perspective. It should be noticed, however, that projection and figures in perspective are not the same. In projection we have reference to the plane of projection, etc., whereas in perspective the thought of the planes on which the corresponding figures lie is absent, and the only necessary condition is that the lines joining corresponding points should be concurrent. It follows therefore that a figure and its projection are in perspective, while two figures in perspective are not necessarily the projection, one of the other.

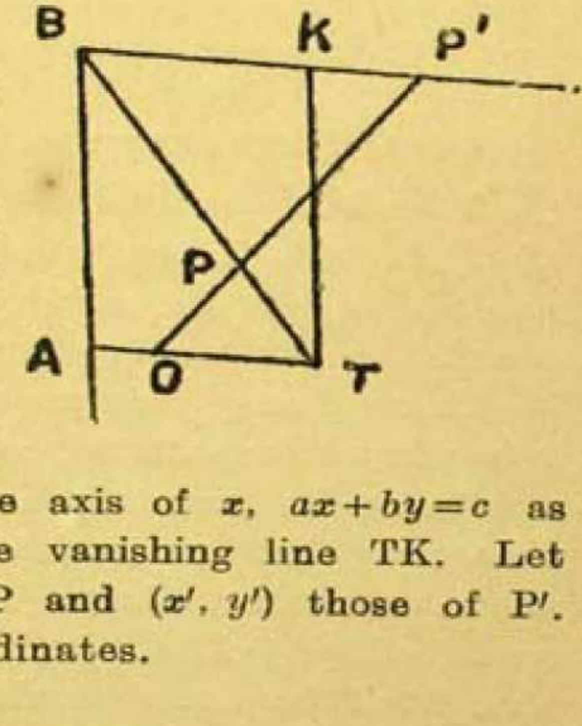
In § 11, suppose that p' turns about its line of intersection l with p, till it coincides with p. Then the two figures, which were originally projections of one another, are now in the *same* plane, while the lines joining corresponding points still pass through a fixed point, and the corresponding lines of the two figures intersect on the

fixed line l. These two figures are then said to be in *plane perspective*, and the line l is called the *axis of perspective*.

The figures may be regarded as plane projection of each other.

14. Analytical Treatment of Plane perspective : *

Let O be the centre and the line AB the axis of perspective and its parallel line TK be the vanishing line. Let any line through O meet TK in T and let TP meet the axis in B. Through B draw BK parallel to OT, meeting OP in P′. Then P′ is the projection of P. The line OT may be any line through O, and the same point P′ will be obtained in any case.



Take O as origin, OT as the axis of x, $ax+by=c$ as the axis AB, $ax+by=c'$ as the vanishing line TK. Let (x, y) be the co-ordinates of P and (x', y') those of P′. Let X, Y be the current co-ordinates.

Then, TP is the line—

$$y\left(X-\frac{c'}{a}\right)=Y\left(x-\frac{c'}{a}\right) \quad \ldots \quad (1)$$

Any line through B is

$$k\left(aX+bY-c\right)+y\left(X-\frac{c'}{a}\right)-Y\left(x-\frac{c'}{a}\right)=0$$

$\therefore$ Since BP′ is parallel to OT, BP′ is the line—

$$Y(ax+by-c')=y(c-c') \quad \ldots \quad (2)$$

* For a detailed account, see Scott, Modern Analytical Geometry, Chapter X.

and OP is the line— $$\frac{X}{x}=\frac{Y}{y} \quad \ldots \quad (3)$$

Hence, equations (2) and (3) will give by their intersection the co-ordinates of P′.

Thus,

$$X=x'=\frac{x(c-c')}{ax+by-c'}, \quad Y=y'=\frac{y(c-c')}{ax+by-c'}$$

and $$x=\frac{x'}{ax'+by'-c+c'}, \quad y=\frac{y'}{ax'+by'-c+c'} \quad (4)$$

If the axis AB is to pass through O, then $c=0$, and the equations of transformation become—

$$x=\frac{x'}{ax'+by'+c'}, \quad y=\frac{y'}{ax'+by'+c'} \quad (5)$$

Ex. 1. Apply to the curve $3x^2+y^2=4$ the transformation in which the line $2x+3y=1$ goes off to infinity.

Ex. 2. Find the transformation which will place the figures

$$x^2+2y^2=1 \quad \text{and} \quad 2y^2+2x=1$$

in perspective positions.

Ex. 3. Apply the above transformation to the curve

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0.$$

15. Theory of Inversion :

If on a radius vector OP, drawn from a fixed origin O to a point P, a point P′ is taken such that the rectangle OP.OP′ is constant $=k^2$, then the point P′ is called the " inverse " of P with respect to a circle with centre O and radius k. It is convenient sometimes to speak of P and P′ as inverses of each other with respect to the point O and the process is called " inversion."

If P traces out a locus C, P′ also traces a corresponding locus C′, which is the inverse of C with respect to the point O. O is called the *origin*, and k the *radius*, of inversion.

The polar equation of the inverse is obtained by putting k^2/r for r in the equation of the original curve.

If O is taken as the origin of a Cartesian system of co-ordinates and (x, y) the co-ordinates of P, then the co-ordinates (x', y') of the inverse point P′ are given by—

$$x' = \frac{k^2 x}{x^2+y^2}, \qquad y' = \frac{k^2 y}{x^2+y^2}.$$

Hence, if $f(x, y) = 0$ is the equation of a curve, the equation of the inverse curve is—

$$f\left(\frac{k^2 x}{x^2+y^2}, \quad \frac{k^2 y}{x^2+y^2}\right) = 0.$$

Thus, the inverse of the straight line $lx + my + n = 0$ is

$$n(x^2+y^2) + k^2(lx+my) = 0,$$

which is evidently a circle through the origin, but the inverse of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

becomes

$$c(x^2+y^2) + 2k^2(gx+fy) + k^4 = 0$$

which is again a circle.

If the line passes through the origin, $n = 0$, and the inverse is the line itself. If the circle passes through the origin, $c = 0$, and its inverse reduces to a straight line.

Hence we see that the inverse of a straight line is a circle, but if the origin lies on the line, it is its own inverse; while the inverse of a circle is a circle, but when the origin lies on the circle, the inverse reduces to a right line.

It follows therefore that the inverse of a system of parallel lines is a system of circles having the same tangent at the origin.

If the line passes through one circular point, its inverse is a line through the other circular point. If P and P′, Q and Q′ are inverse points on two inverse curves, we have $OP.OP' = OQ.OQ'$, so that PP′Q′Q are concyclic, and consequently $\angle OPQ = \angle OQ'P'$. If now P becomes consecutive to Q, and P′ to Q′, PQ and P′Q′ become tangents at P and P′ on the inverse curves respectively, and they make supplementary angles with the radius OPP′.

From this it may be shewn that two curves cut at the same angle as their inverses.

If the point P describes a curve in space, not necessarily a plane curve, then P′ is said to be the inverse of P with respect to a *sphere* with O as centre.

We shall have occasion in a subsequent Chapter to discuss more fully the general theory of inversion.

Ex. 1. A circle is inverted into a line. Prove that this line is the radical axis of the circle and the circle of inversion.

Ex. 2. A system of intersecting coaxal circles can be inverted into concurrent straight lines.

Ex. 3. What is the inverse of the pair of isotropic lines given by

$$(x-a)^2 + (y-b)^2 = 0$$

with respect to the origin?

Ex. 4. The angle between a circle and its inverse is bisected by the circle of inversion.

16. Reciprocation :

Suppose a fixed conic C is given. If we find the pole P of any tangent p to a given curve S with regard to C,

then the locus of P will be a curve s, which is called the *polar curve* of S with regard to C, and C is called the *auxiliary conic*; P is said to correspond to p, consequently every point of s corresponds to some tangent to S.

Now if p, p' are two tangents to S, their corresponding points P, P′ are points on s, and the point of intersection of p, p' is the pole of the line PP′. Now, when p, p' are consecutive tangents to S, their intersection is a point of S, P and P′ become consecutive points on s, and the line PP′ becomes a tangent to s. Hence, if any tangent to S corresponds to a point on s, the point of contact of that tangent to S will correspond to the tangent at the point on s.

Thus the relation between the curves S and s is reciprocal, *i.e.*, the curve S may be generated from s in precisely the same manner that s is generated from S. Hence the curves are called *reciprocal polars*, and the process is called "*reciprocation.*"

Analytical and other aspects of the theory will be discussed in a separate chapter.

Ex. 1. The polar reciprocal of a circle with regard to another is a conic having the origin for focus.

Ex. 2. A system of non-intersecting coaxal circles can be reciprocated into confocal conics.

Ex. 3. Conjugate points of one conic reciprocate into conjugate lines of the reciprocal, and a self-conjugate triangle reciprocates into a self-conjugate triangle.

CHAPTER II

Plane Algebraic Curves

Sec. I.—General Properties.

17. Notion of Algebraic Curves:

If on the plane of the paper we draw a line of any form, it is called a "curve." It is theoretically possible always to represent this curve, described according to certain laws, by means of a definite analytical equation in any system of co-ordinates, or at least, by means of a Fourier Series. We propose to investigate the properties of a curve represented by means of an equation $F(x, y)=0$ in Cartesian co-ordinates, and in doing so, we shall have first to study the nature of the function $F(x, y)$.

In former days, functions were divided into two separate classes—(1) algebraic, and (2) transcendental, and the curves represented by them were accordingly divided into two classes *—"*algebraic curves*" and "*transcendental curves.*" This classification was chiefly based on the use of Cartesian co-ordinates and therefore does not hold when other systems are used. But this is of little importance compared with the great advantage we derive in studying the properties of algebraic curves in the light of the modern theory of functions. For functions both of whose ranges are real numbers, a graphical representation was devised by Descartes. The older Mathematicians held that a function simply meant a single formula, at first usually a power of the variable, but afterwards it was regarded as defined by any analytical expression, and was extended by Euler † to include the case in which the function is given implicitly as a formal relation between

* Descartes, *Geometrie* (Leyden, 1637).

† *Memoires de l'Acad des Sc.*, T. 4 and 5.

the two variables. The arbitrary nature of a function given by a graph was recognised by Fourier. In the modern theory of functions, it is held that a regular function can be completely defined by means of a graph drawn in the finite and continuous domain of the independent variable. This simply depends upon geometrical intuition. A curve thus drawn is indistinguishable by perception from a sufficiently great number of discrete points. But a graph can only be regarded as an approximate representation of a function, and all that is really given by the graph consists of more or less arithmetically approximate values of the ordinates at those points of the x-axis at which we are able to measure them.

In order that a curve may be drawn really to define a function, certain laws must be formulated, by means of which the values of the ordinates can be formally determined at all points of the x-axis. These considerations led to the classification of functions in the modern theory, according as they possess various special properties, namely, continuity, differentiability, integrability, etc., throughout the domain of the independent variable, or at or near special points in that domain. The modern theory of functions says that the equation $F(x, y)=0$ cannot in general represent a curve; it can do so, if y can be expressed as a regular function $f(x)$ of x, *i.e.*, if $f(x)$ is a continuous, finite and unlimitedly differentiable function expansible in Taylor's series. It is only by the combination of these conditions that $y=f(x)$ can represent a curve.

18. Representation of Functions:

Each algebraic function is put in the following rational and integral form:—

$$F(x, y) \equiv \sum_{r=0}^{m}\sum_{s=0}^{n} a_{r,s} x^r y^s.$$

All other functions which cannot be put into this form are called "transcendental" functions. A class of functions

of the form $y = x^{\sqrt{2}}$ is called by Leibnitz " interscendental " functions. These functions involve variables with exponents not commensurable with any rational number.

Parametric representation of functions is often very useful in the form $x = \phi(t)$, $y = \psi(t)$, from which the implicit form of the function is obtained by eliminating the variable t.

19. From what has been said above it follows that under certain conditions $F(x, y) = 0$ is the equation of an algebraic curve, while the curve itself is the geometric representation of the function $F(x, y)$. Thus, in general, a curve of the nth order is defined as the geometric representation of a function which is of the nth degree in the variables. Different special names have been devised to denote curves of different orders; for instance, a curve of the third order is called a " cubic," of the fourth order a " quartic," and in general, one of the nth order is called an n-ic.

If the quantic $F(x, y)$ is *irreducible*, *i.e.*, if it cannot be broken into two or more rational and integral factors of lower dimensions, the curve is called a " *proper* or *non-degenerate curve* " of the nth order. But if $F(x, y)$ breaks up into two or more rational and integral factors of lower orders, the curve is called an " *improper* " or " *degenerate* " curve. Thus, two right lines together form a *degenerate conic*; a conic and a line, or three right lines constitute a *degenerate cubic*; two conics constitute a " *degenerate quartic*, " and so on.

20. The most general equation of the nth degree in two variables may be written as :—

$$\begin{array}{c} a \\ + bx + cy \\ + dx^2 + exy + fy^2 \\ + gx^3 + hx^2y + ixy^2 + jy^3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ + px^n + qx^{n-1}y + \ldots\ldots + rxy^{n-1} + sy^n = 0 \end{array} \quad \ldots \quad (1)$$

The above equation will sometimes be written in the symbolic form

$$u_o + u_1 + u_2 + u_3 + \ldots \quad + u_r + \ldots \quad + u_n = 0 \quad \ldots \quad (2)$$

where u_o is a constant and u_r is a binary quantic (r-ic) in x and y.

The equation (2) may be made homogeneous by introducing a third variable z, and the general equation of a curve of order n in homogeneous co-ordinates may be written as:

$$u_o z^n + u_1 z^{n-1} + u_2 z^{n-2} + \ldots + u_r z^{n-r} + \ldots + u_n = 0 \;\ldots\; (3)$$

From these equations it is seen that the number of terms in u_{r-1} is r, and that in u_r is obviously one more than in u_{r-1}, and is therefore equal to $r+1$. Thus the total number of terms in each of these equations is equal to $1+2+3+\ldots\ldots+n+(n+1)$, *i.e.*, $\frac{1}{2}(n+1)(n+2)$.

The number of independent constants in each of these equations is equal to one less than the number of terms it contains, for the generality of the equation remains unchanged, and consequently it represents the same curve, if we divide the whole expression by a constant. Thus, dividing the equation by u_o, and substituting new constants for the ratios of old ones to u_o, the number of constants is reduced by *one*. Hence the general equation of the nth degree contains only $\{\frac{1}{2}(n+1)(n+2)-1\}$ or $\frac{1}{2}n(n+3)$ *independent* constants and can therefore be made to satisfy the same number of conditions, and no more.

21. Number of Points determining a Curve:

Since the general equation contains $\frac{1}{2}(n+3)$ independent constants, the same number of conditions are required to determine the co-efficients uniquely. Hence $\frac{1}{2}n(n+3)$ conditions are required to determine a curve of order n; for instance, the curve may be made to pass through $\frac{1}{2}n(n+3)$ given points. Thus, a curve of order n

is uniquely determined, if $\frac{1}{2}n(n+3)$ points on it are given. The co-ordinates of each of the given points satisfy the equation giving a linear relation between the constants. We obtain in this way altogether $\frac{1}{2}n(n+3)$ equations to determine the same number of unknown quantities, *i. e.*, the co-efficients. There is only one set of solutions, and the equation is uniquely determined, and consequently the curve also. Hence it follows that

Through $\frac{1}{2}n(n+3)$ given points, only one curve of the nth degree can be described.

Thus a conic is determined by five points, a cubic by nine points, a quartic by fourteen points, and so on.

22. Proper and degenerate Curves:

We have seen that $\frac{1}{2}n(n+3)$ points will determine a curve of order n uniquely; but they do not in all cases determine a proper or non-degenerate curve. For instance, five points determine a conic, but if three of them lie on a right line, the conic determined by them consists of two right lines. Nine points determine a cubic, but if three of them lie on a right line, the cubic consists of this line and a conic through the remaining six points. Thus we see that the $\frac{1}{2}n(n+3)$ given points do not *always* determine a non-degenerate curve of the nth degree.

The necessary and sufficient condition that they can determine a non-degenerate curve uniquely is that the points must all be "*independent*," *i. e.*, no group of them should lie on a curve of order lower than they can usually determine. This follows from the fact that the linear relations which determine the co-efficients must all be independent, and their number must be exactly $\frac{1}{2}n(n+3)$, neither more nor less. For, if two or more of the co-efficients are connected by other relations, the co-efficients are not independent, but they satisfy identical relations. In this case some of the points lie on one or more other

curves of lower orders and the curve of the nth order is not a non-degenerate curve, but consists of two or more curves of lower orders. If the number of these linear relations be less than $\frac{1}{2}n(n+3)$, which is the case when one or more of them can be deduced from the others by algebraic operations, the co-efficients cannot be uniquely determined and the equation contains one or more indeterminate co-efficients. In this case an infinite number of curves can be described through the points.

Thus we see that in general $\frac{1}{2}n(n+3)$ arbitrary given points determine a "non-degenerate curve," and that uniquely.

23. Intersections of Curves:

Two curves of orders m and n respectively intersect in general in mn points.

Let

$$V_m \equiv a + bx + cy + dx^2 + \dots \quad = 0 \quad \dots \quad (1)$$

and $$V_n \equiv a' + b'x + c'y + d'x^2 + \dots \quad = 0 \quad \dots \quad (2)$$

be the equations of two curves of orders m and n respectively.

Then, at the common points of intersection of the two curves both the equations (1) and (2) must be simultaneously satisfied. Hence if we eliminate either of the variables x or y between them, the resulting equation will be satisfied by the values of the other variable at the common points of intersection. Thus, if y be eliminated between (1) and (2), the resulting equation in x—the eliminant—will be of degree mn and will determine the abscissæ of the points of intersection of the two curves.

The resulting equation will have either all its roots real, or if there are imaginary roots, they will occur in pairs. Therefore, if there are $2k$ imaginary roots, the number of real intersections will be $mn—2k$, where k is zero or a positive integer.

Thus a straight line intersects a curve of the nth degree in n points, and when n is odd, one of these must be real. It is to be noted, however, that every line cannot meet an n-*ic* in n real points.

A conic intersects an n-*ic* in $2n$ points, of which all may be imaginary, or an even number real. A cubic intersects it in $3n$ points, and so on.

24. Although two curves of orders m and n respectively intersect in mn points, but yet mn points taken arbitrarily on a curve of order n will not be the intersections of two such curves. In fact, a certain number of these points being given, the rest will be determined.

We shall now consider how many of these mn points can be arbitrarily chosen on the n-*ic*, in order that the remaining points may thereby be completely determined. There are three cases to be considered :—

Case I. If $m<n$, let $\Phi=0$ be a curve of order n. Then $\frac{1}{2}m(m+3)$ points taken on $\Phi=0$ will completely determine a curve $\Psi=0$, of order m, which will intersect Φ not only in these $\frac{1}{2}m(m+3)$ assumed points but also in $mn-\frac{1}{2}m(m+3)$ other points.

Case II. If $m \geq n$, we may consider a degenerate curve $\Psi \equiv \Phi . \Psi'=0$, of order m, consisting of Φ and another curve Ψ', of order $m-n$. Then the $\frac{1}{2}(m-n+1)(m-n+2)$ co-efficients of Ψ' are at our disposal and can therefore be so chosen that an equal number of co-efficients in Ψ vanish. ψ will then involve only

$$\tfrac{1}{2}m(m+3)-\tfrac{1}{2}(m-n+1)(m-n+2)=mn-\tfrac{1}{2}(n-1)(n-2)$$

constants, and therefore the same number of points will determine the curve $\Psi=0$. This is also true if Ψ be a proper curve.

Hence, if $mn-\frac{1}{2}(n-1)(n-2)$ points are given on the curve Φ, another curve Ψ of degree m drawn through them will intersect Φ in $\frac{1}{2}(n-1)(n-2)$ other fixed points.

Thus, $\frac{1}{2}m(m+3)$ of the mn points of intersection can

be chosen arbitrarily, when $m<n$, so that $mn-\frac{1}{2}m(m+3)$ points are thereby determined.

If $m \geqq n$, $mn-\frac{1}{2}(n-1)(n-2)$ points can be assumed arbitrarily, and then the remaining $\frac{1}{2}(n-1)(n-2)$ points are determined. For $m=n-1$ or $m=n-2$, the two numbers $mn-\frac{1}{2}m(m+3)$ and $\frac{1}{2}(n-1)(n-2)$ are the same.

Hence we can say that, when $m \geqq n$, or $m=n-1$ or $n-2$, $mn-\frac{1}{2}(n-1)(n-2)$ points can be chosen arbitrarily and the remaining $\frac{1}{2}(n-1)(n-2)$ points are thereby determined.*

25. If $m=n$, the above theorem becomes—*If the n^2 points of intersection of two curves of the nth degree, $n^2-\frac{1}{2}(n-1)(n-2)$, i.e., $\frac{1}{2}n(n+3)-1$ are given, the remaining $\frac{1}{2}(n-1)(n-2)$ are also determined.*

Or, in other words,

All curves of the nth degree which pass through $\frac{1}{2}n(n+3)-1$ points pass also through $\frac{1}{2}(n-1)(n-2)$ other fixed points.

This theorem can be proved independently as follows:—

Let $\Phi=0$ and $\Psi=0$ be any two particular curves of the nth degree passing through $\frac{1}{2}n(n+3)-1$ given points. Then the general equation of a curve of the same degree through these points is $\Phi+k\Psi=0$, where k is any arbitrary parameter. The equation is evidently satisfied by $\Phi=0$ and $\Psi=0$ for all values of k, and consequently the curve passes through all the n^2 intersections of Φ and Ψ, whatever be the value of k. Hence this curve passes not only through the $\frac{1}{2}n(n+3)-1$ given points, but also through the remaining $n^2-\frac{1}{2}n(n+3)+1$ or $\frac{1}{2}(n-1)(n-2)$ points of intersection of Φ and Ψ.

In fact, through the intersections of Φ and Ψ, an infinity of curves of the nth degree can be described, and

* The theorem also holds if the curve of the nth degree breaks up into two curves of the kth and n-kth degrees respectively, when the curve of the mth degree does not pass through all the intersections of the two curves—see Zeuthen—*Sur la determination d'une courbe algebraique par des points doneé's*—Math. Ann. Bd. 31 (1888).

any particular member will be determined by the condition that it passes through any other point. But if this last-mentioned point is one of the common points of Φ and Ψ, the value of k is indeterminate. It may be observed, however, that if $n>3$, the above theorem is not *always true*.

Ex. 1. Straight lines are drawn through each of n collinear points on an n-*ic*. Shew that these lines meet the n-*ic* again in points on an $(n-1)$-*ic*.

Ex. 2. The tangents at four collinear points of a quartic meet the curve again in eight points on a conic.

Ex. 3. All quartic curves drawn through 3 given points will pass through three other fixed points.

Does this hold when the quartics are degenerate, each consisting of four right lines?

Note: This is the so-called *Paradox of Cramer*. Euler had already noticed that two curves of the nth degree intersect in more points than are sufficient to determine any of them.* Cramer † fully discussed this paradox but could assign no reason for this. The first notion of systems of curves was given by Lamé, and then Plücker in a note to "Entwicklungen" (Vol. 1, p. 228) gave the above theorem. The algebraic aspect of the question was discussed in the works of Jacobi (Journal of Math., Vol. XV, 1841) and Plücker (*Ibid*, Vol. XVI, 1842).‡

26. If $m>n$, the theorem of § 24 may also be stated in the following form:—*All curves of the mth degree which pass through* $nm-\frac{1}{2}(n-1)(n-2)$ *points on a curve of order n* $(m>n)$, *pass also through* $\frac{1}{2}(n-1)(n-2)$ *other fixed points on this curve*.

* Euler—A Memoir in the Berlin Transactions for 1748—" On an apparent contradiction in the Theory of Curves."

† Cramer—Introduction à l'Analyse des lignes courbes algébriques (1750).

‡ Plücker's *Algebraischen Curven, Einleitung*; and also a Memoir by Prof. Cayley—Cambridge Math. Journal, Vol. III, p. 211.

Let $\Phi=0$ be the given curve of the nth degree and $f=0$ be a curve of order $m-n$, determined by $\frac{1}{2}(m-n)(m-n+3)$ assumed points. Now consider the curve $\Psi\equiv f.\Phi=0$. This curve passes through

$$nm-\tfrac{1}{2}(n-1)(n-2)+\tfrac{1}{2}(m-n)(m-n+3), \text{ or } \tfrac{1}{2}m(m+3)-1$$

points and therefore, by the preceding theorem, it passes through $\frac{1}{2}(m-1)(m-2)$ other fixed points. Of these latter points some will lie on $\Phi=0$ and some on $f=0$; and as many will lie on Φ as will make the total number of points to nm, and as many will lie on f as will make the total number to $m(m-n)$. Thus $\frac{1}{2}(n-1)(n-2)$ of these $\frac{1}{2}(m-1)(m-2)$ fixed points must lie on the curve $\Phi=0$.

Ex. 1. Every quartic curve drawn through eleven fixed points on a cubic passes through another fixed point on the cubic.

This follows from the present theorem by putting $m=4$, $n=3$. We can prove this directly as follows :—

Every quartic curve through these eleven points and two other assumed points passes through three other fixed points. Now the given cubic and the line determined by the two assumed points make up a quartic through these thirteen points. Therefore, this system must pass through three other fixed points, of which only one can lie on the cubic, for otherwise a quartic through the points would meet the cubic in more than 12 points.

Ex. 2. A sextic Φ intersects a curve Ψ of the eighth order in 48 points. But in order to determine $\Psi=0$, 44 points are required. Of these points if 38 are given, the remaining 10 are determined. A system of curves Ψ through these 38 points on Φ will pass through 10 other fixed points on the same.

27. We have seen that if a curve of order m passes through $nm-\frac{1}{2}(n-1)(n-2)$ points on a curve of order n, it passes also through $\frac{1}{2}(n-1)(n-2)$ other fixed points on the same. Hence it follows that, of the $\frac{1}{2}m(m+3)$ points required to determine a curve of order m, if more than $nm-\frac{1}{2}(n-1)(n-2)$ lie on a curve of order n $(m \geqslant n)$, the curve cannot be a proper curve. For,

suppose one more of the points lies on the curve of the nth degree; in that case the curve of the mth degree would meet the curve of the nth degree not only in $nm - \frac{1}{2}(n-1)(n-2)+1$ points, but also in $\frac{1}{2}(n-1)(n-2)$ other points, thus making the total number to $nm+1$, which is impossible. If, however, the additional point be one of the $\frac{1}{2}(n-1)(n-2)$ fixed points, the curve cannot be uniquely determined; for the number of independent points is then

$$nm - \tfrac{1}{2}(n-1)(n-2) + \{\tfrac{1}{2}m(m+3) - nm + \tfrac{1}{2}(n-1)(n-2) - 1\},$$

i.e.,
$$\tfrac{1}{2}m(m+3) - 1.$$

Hence we obtain the theorem—

Of the points determining a proper curve of the mth degree, the greatest number which can lie on a lower curve of order n is $nm - \frac{1}{2}(n-1)(n-2)$.

28. Chasles' Theorem :*

If a curve of the third order pass through eight of the points of intersection of two curves of the third order, it passes through the ninth point of intersection.

This in fact is a particular case of the more general theorem, given in § 25. Chasles has given the following proof :—

Let $U=0$ and $V=0$ be any two curves of the third order drawn through eight given points. The general equation of a curve of the third order through these eight points is of the form $U=kV$, where k is indeterminate. If this curve passes through a ninth point (x', y'), we must have $U'=kV'$, where U' and V' are the values of U and V when x' and y' are substituted for x and y in them. Then the equation of the cubic becomes $UV'-U'V=0$. But this equation is satisfied by all the nine points of intersection of the two curves U and V, and therefore this

* Chasles—*Memoires de Bruxelles*, Vol. XI, p. 149.

curve passes not only through the eight of these points of intersection, but also through the ninth.

Chasles has used this theorem to demonstrate Pascal's theorem on Hexagon, namely,—*the three intersections of the opposite sides of any hexagon inscribed in a conic section are in one right line.*

Consider a hexagon inscribed in a conic section. The aggregate of three alternate sides may be regarded as forming a curve of the third order, and that of the remaining sides a second curve of the same order. These two intersect in nine points, namely, the six angular points of the hexagon and the three points which are the intersections of the pairs of opposite sides. Now the conic section and the line joining two of these intersections constitute a curve of the third order passing through eight of those nine points. Therefore this passes also through the ninth point by the present theorem, *i.e.*, the line passes through the remaining point, since the third intersection cannot lie on the conic.

29. The following important theorem* was given by M. Gergonne :—

If of the n^2 points of intersection of two curves of order n, nm lie on a curve of order $m (m<n)$, the remaining $n(n-m)$ will lie on a curve of order $n-m$.

Let $U=0$ and $V=0$ be any two curves of the nth degree intersecting in n^2 points, and let $\Phi=0$ be a curve of the mth degree through nm of these points. Select a curve $\Psi=0$, of order $n-m$, determined by $\frac{1}{2}(n-m)(n-m+3)$ of the remaining points, and consider the curve $f\equiv\Phi.\ \Psi=0$ of the nth degree which passes through—

$nm+\frac{1}{2}(n-m)(n-m+3)$ or $\{\frac{1}{2}n(n+3)-1+\frac{1}{2}(m-1)(m-2)\}$

of the points of intersection of $U=0$ and $V=0$. But this number is not less than $\frac{1}{2}n(n+3)-1$. Therefore the curve

* Annales, Vol. XVII, p. 220.

f passes through all the n^2 intersections of U and V by § 25. But f consists of an m-ic Φ, which already passes through mn of these n^2 points, and cannot therefore pass through any more. Consequently, the other curve Ψ of order $(n-m)$ passes through the remaining n^2-mn, or, $n(n-m)$ points.

It will be observed that f passes through the remaining

$$\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}(m-1)(m-2)$$

intersections of U and V. But these latter points cannot lie on Φ, which already meets the system of the nth degree curves in nm points. Hence they lie on $\Psi=0$, which therefore passes through—

$$\tfrac{1}{2}(n-m)(n-m+3)+\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}(m-1)(m-2),$$

$$i.e., \quad n(n-m) \text{ points.}$$

Ex. 1. If of the nine intersections of two cubics, six lie on a conic, the remaining three are collinear. (Put $n=3$, $m=2$.)

Ex. 2. If of the sixteen intersections of two quartic curves, eight lie on a conic, the remaining eight lie on another conic. ($n=4$, $m=2$.)

Ex. 3. Any quartic through the intersections of a conic and a quartic meets the quartic again in eight points on a conic. ($n=4$, $m=2$.)

Ex. 4. Any quartic through the intersections of a cubic and a quartic meets the quartic again in four collinear points. ($n=4$, $m=3$.)

Ex. 5. If a polygon of $2n$ sides be inscribed in a conic, the $n(n-2)$ points, where each odd side intersects the non-adjacent even sides, will lie on a curve of the $(n-2)$th degree.

This is a particular case of the present theorem, when $m=2$, and U and V are degenerate curves, each consisting of n right lines.

The product of all the odd sides may be regarded as one system of the nth degree, and the product of the even sides another. These two systems intersect in n^2 points, *namely*, the $2n$ vertices of the polygon and the remaining $n(n-2)$ points, which are the intersections of the n odd sides with the $(n-2)$ non-adjacent even sides of each, since each odd side has two adjacent even sides. Now, the $2n$ vertices lie on a conic, and therefore by the present theorem, the remaining $n(n-2)$ points lie on a curve of the $(n-2)$th degree.

Pascal's theorem on hexagon is a particular case of this, when $n=3$.

30. Cayley's Theorem: Prof. Cayley has given the following general theorem,* of which the theorem of § 26 is a particular case :—

If a curve of the rth order (r not less than m or n, not greater than $m+n-3$) pass through

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of the points of intersection of two curves of the mth and nth orders respectively, it passes through the remaining

$$\tfrac{1}{2}(m+n-r-1)(m+n-r-2) \textit{ points of intersection.}$$

Let $U=0$ and $V=0$ be any two curves of orders m and n respectively.

Select a group of $\frac{1}{2}(r-m)(r-m+3)$ arbitrary points on V and describe a curve $\Phi=0$ of order $(r-m)$ through them.

Again, select a group of $\frac{1}{2}(r-n)(r-n+3)$ points on U and describe a curve $\Psi=0$ of order $r-n$ through them.

Now consider the two curves of order r, namely,

$$f\equiv\Phi.U=0 \quad\text{and}\quad f'\equiv\Psi.V=0.$$

The curve f passes through $\frac{1}{2}(r-m)(r-m+3)$ points on V, $\frac{1}{2}(r-n)(r-n+3)$ points on U, and all the intersections of U and V, *i.e.*, through

$$\{\tfrac{1}{2}(r-m)(r-m+3)+mn+\tfrac{1}{2}(r-n)(r-n+3)\},$$

or $\{\frac{1}{2}r(r+3)-1\}+\frac{1}{2}(m+n-r-1)(m+n-r-2)$ points.

Similarly, the curve f' passes through the same

$$\{\tfrac{1}{2}r(r+3)-1\}+\tfrac{1}{2}(m+n-r-1)(m+n-r-2) \text{ points.}$$

Hence, if we select a group of

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

points of intersection of U and V, then the curves f

* Cayley—Collected Papers, Vol. I, No. 5, p. 25.

and f' will pass through $\frac{1}{2}r(r+3)-1$ points. Therefore, every curve of the rth degree which passes through these $\frac{1}{2}r(r+3)-1$ points will pass through all the intersections of f and f', and consequently through all the mn intersections of U and V. But

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of these mn intersections lie on the curve. Therefore it passes through the remaining $\frac{1}{2}(m+n-r-1)(m+n-r-2)$ intersections.

Note: It will be observed that r cannot be less than m or n; it can at the most be equal to the greater of m and n. In order that the curve Φ may be distinct from and does not include as part of itself, the curve V, $r-m$ must be less than n, *i.e.*, r must be less than $m+n$. If $r=m+n-1$ or $m+n-2$, the theorem is meaningless. Hence, r must not be greater than $m+n-3$. See Bacharach, Math. Annalen, Vol. XXVI (1886), pp. 275-299.

If $m=n=r$, we obtain the theorem of § 25.

Ex. 1. All quartics which pass through eleven points of intersection of a cubic and a quartic also pass through the remaining intersection.

[Put $r=m=4$ and $n=3$.]

Ex. 2. If of the twelve intersections of a cubic and a quartic, six lie on a conic, the other six will also lie on another conic and the four points in which the two conics meet the quartic again are collinear.

Consider the quartic consisting of the given conic and another determined by five of the remaining six points of intersection. This quartic therefore passes through eleven of the points of intersection of the given cubic and the quartic. Therefore it passes through the remaining intersection, which must lie on the second conic, since the given conic meets the cubic already in six points. Hence the remaining six points lie on a conic.

Again, suppose that the first conic intersects the quartic in the two points A and B, and let the line AB meet the quartic in two other points C and D. Now consider the quartic consisting of the cubic and the line AB. This intersects the given quartic in 16 points.

But the quartic consisting of the two conics passes through fourteen of these points. Therefore it must pass through the remaining two points C and D. Consequently, C and D lie on the second conic. Hence the conics intersect the quartic in four collinear points.

This follows from the present theorem, as a particular case, by putting $m=4$, $n=3$, and $r=4$.

Ex. 3. A curve of the fifth degree which passes through fifteen of the intersections of two quartics passes also through the remaining intersection.

This follows from the present theorem by putting $r=5$, $m=n=4$.

Ex. 4. The tangents at the six intersections of a conic and a cubic meet the cubic again in six points on a conic.

Ex. 5. Show that a quartic drawn through the intersections of two cubics meets either cubic again in three collinear points.

Examples 3 and 4 of § 29 can be deduced from the present theorem.

Sec. II.—Theory of Residuation.

31. We have seen (§ 26,) that if of the mn intersections of two curves of the mth and nth degrees ($m>n$),

$$mn-\tfrac{1}{2}(n-1)(n-2)$$

are given, the remaining $\frac{1}{2}(n-1)(n-2)$ are determined. Hence we see that the two groups of points are not independent, but together form the complete intersection of two curves of orders m and n respectively. These considerations lead to the development of a theory of groups of points on a curve, the principles of which are contained in a paper by Noether.* The principles of the theory are illustrated in the following theorem:

If a group of $n(l+m)$ points on a curve C_n of order n forms the complete intersection of C_n with a curve C_{l+m} of order $(l+m)$, and nl of these points form the complete intersection of C_n with a curve C_l of order l,

* Noether—"Ueber einen Satz aus der Theorie der Algebraischen Functionen"—Math. Ann. Bd. 6, 1872, and "Zum Beweise des Satzes, etc.," Math. Ann. Bd. 40, 1892, p. 140.

then the remaining nm points form the complete intersection of C_n with a curve C_m of order m.

The theorem of § 29 is only a particular form of this theorem, when $l+m=n$.

The truth of the present theorem has led to the development of a theory of sets of points on curves known as the *theory of residuation*,* which is at times useful in discussing intersections of curves. The following definitions are useful :

(1) The curve C_n is called the *basis-curve*.

(2) Two groups of points are said to be *residual* of each other, when they together form the complete intersections of any curve with the basis-curve.

(3) Any two groups of points on a basis-curve which have the same residual group are said to be *co-residual* to each other.

(4) If a group is the complete intersection of a curve with the basis-curve, it is said to have a *zero residual*.

32. In the light of these definitions the above theorem may be stated in the following form :—

If a group of $n(l+m)$ points on a curve of order n has a zero residual and nl of these points have also a zero residual, then the remaining nm points have a zero residual.

Consider the curve $C_{l+m} \equiv C'_{l+m-n}.C_n + C'_m C_l = 0$, $l+m>n$, which is of order $(l+m)$ and passes through the intersections of C_l and C_n. This equation is satisfied for all points where $C_n=0$ and $C'_m C_l=0$. Now the curves C_{l+m} and C_n intersect in $n(l+m)$ points, of which ln lie on the curve C_l. Therefore the remaining nm points lie on the curve C'_m of order m.

* Sylvester—Coll. Works, Vol. III, pp. 317 and 352.

If $l+m=n$, then we have simply to put—

$$C'_{l+m-n}=C'_o=\text{a constant.}$$

The proof can be equally applied to the case when $l+m<n$. We have to put $C'_{l+m-n}=C_{-k}=0$, and C_{l+m} becomes simply $C'_m C_l$. In this case the curve C_{l+m} cannot be a non-degenerate curve, but consists of two curves C_l and C'_m, of orders l and m respectively.

If we denote the two groups of nl and nm points by [L] and [M] respectively and express the fact that a group [P] has a zero residual by the symbolic equation $[P]=0$, the above theorem may be stated as follows:—

If $[L+M]=0$ and $[L]=0$, then also $[M]=0$.

This leads us to the supposition that the residual equations obey the general laws of addition and subtraction. In fact, the above theorem says that of the three equations $[L+M]=0$, $[L]=0$, $[M]=0$, any one can be deduced from the other two. These lead to the following theorems on addition and subtraction in the residual theory.

33. Addition Theorem:

If $[L]=0$ and $[M]=0$, then $[L+M]=0$.

This, in the language of geometry, may be stated as:—

If two curves C_l and C_m, of orders l and m respectively intersect a curve C_n of order n in the groups of points nl and nm, then a curve C_{l+m} of order $l+m$ can be drawn to intersect C_n in these $nl+nm$ points.

Let the groups of nl and nm points be denoted by [L] and [M] respectively. We have to consider the three cases, according as $l+m \gtreqless n$.

Case I. When $l+m>n$, $l+m-n$ is a positive quantity.

Consider the equation $C_{l+m}=C'_{l+m-n}C_n+C_lC_m=0$, which represents a curve of order $l+m$.

This equation is satisfied when $C_l=0$ and $C_n=0$, and therefore the curve passes through the nl intersections of C_l and C_n. Similarly, the equation is satisfied when $C_m=0$, and $C_n=0$, and therefore the curve passes through the nm points of intersection of C_m and C_n. Hence the curve C_{l+m} passes through all the points of the two groups [L] and [M], and since the number of points in the two groups is $n(l+m)$, the points of the group [L + M] are the complete intersections of C_{l+m} with C_n.

Hence we must have $[L+M]=0$.

Case II. If $l+m=n$, $l+m-n=0$ and C'_{l+m-n} is to be put equal to a constant, and the same argument holds.

Case III. If $l+m<n$, $l+m-n$ is a negative quantity. In this case the curve C_{l+m} cannot be a proper curve of degree $l+m$. For, C_{l+m} intersects C_l in $l(l+m)$ points, and therefore this must be greater than or equal to the number of points in [L], *i.e.*, $l(l+m)\geq nl$, which is impossible. Thus, C_{l+m} cannot be a proper curve. In fact, we have to take $C_{l+m}\equiv C_l.C_m$, which therefore passes through all the points of the groups [L] and [M].

$$\therefore \quad [L+M]=0.$$

34. Subtraction Theorem:

If $[L+M]=0$ and $[L]=0$, then $[M]=0$.

Let C_n be the basis-curve and let C_{l+m} and C_l intersect the basis-curve C_n in the groups of points [L + M] and [L] respectively.

Let $l+m\geq n$; and consider the equation—

$$C_{l+m}+C_n.C'_{l+m-n}=0.$$

This represents a curve of order $l+m$. The equation is satisfied when $C_{l+m}=0$ and $C_n=0$, and therefore denotes a curve passing through the group of points $[L+M]$, in which C_n intersects C_{l+m}. For different forms of C'_{l+m-n}, this equation represents different curves of order $l+m$ through the same group $[L+M]$. Now $C_l.C'_m$ may be regarded as a system of the $(l+m)$th degree through the same points. But C_l intersects C_n in nl points, and therefore the remaining mn points must lie on C'_m, which is of degree m, and consequently the mn points $[M]$ are the complete intersections of C'_m with C_n,

$$\therefore \quad [M]=0.$$

35. Multiplication Theorem:

If $[L]=0$, then $[kL]=0$, where k is a positive integer.

Let C_n be the basis-curve, and let C_l, a curve of order l, cut C_n in the zero residual group of points $[L]$, namely,

$$[L_1, \ L_2, \ \ldots \ L_{nl}]=0.$$

Consider another curve C'_l contiguous to C_l. This cuts C_n in a group of points

$$[L'_1, \ L'_2, \ L'_3, \ \ldots \ L'_{nl}],$$

which is also a zero residual.

Hence, by the addition theorem, we have—

$$[L_1, L'_1, L_2, L'_2, \ldots L_{nl}, L'_{nl}]=0,$$

or $$[L_1+L'_1, L_2+L'_2, \ldots L_{nl}+L'_{nl}]=0.$$

But since C_l and C'_l are contiguous curves, the points

$$(L_1, L'_1), (L_2, L'_2), \text{ etc.}$$

are contiguous in pairs, so that ultimately L_1 and L'_1, etc., are coincident. We therefore obtain—

$$[2L_1, 2L_2, 2L_3 \ldots 2L_{nl}]=0, \quad \textit{i.e.}, \quad [2L]=0.$$

Repeating the same process for another series of contiguous points, we obtain $[3L]=0$.

Continuing in this way k times, we finally obtain $[kL]=0$, where k is any positive integer.

Note. It is to be understood in this theorem that when a point is multiplied by a positive integer, consecutive points on the curve are taken into account. *There is no division theorem in theory of Residuation.*

Ex. 1. If of the nine intersections of two cubics, six lie on a conic, he remaining three lie on a right line.

Let the six points which lie on the conic be denoted by $[L]$ and the remaining three by $[M]$.

Then $[L+M]=0$, and also $[L]=0$ $\therefore [M]=0$.

i.e., the three points $[M]$ are the complete intersections of the basis cubic with another which must evidently be a right line.

Ex. 2. If eight of the sixteen intersections of two quartics lie on a conic, the remaining eight lie on another conic.

Let the eight points on the conic be denoted by $[L]$ and the remaining eight by $[M]$. Then $[L+M]=0$, and also $[L]=0$,

$$\therefore \quad [M]=0.$$

i.e., the eight points $[M]$ are the complete intersections of the basis quartic with a curve of degree m, such that $4m=8$, and consequently $m=2$, *i.e.*, the eight points lie on a conic.

36. Residual Equations:

If we have the two residual equations $[L+N]=0$ and $[M+N]=0$, when L, M, N are groups of points on a basis curve, then we obtain, by the subtraction theorem, the formula $[L-M]=0$, or, $[L]=[M]$, which signifies that the two groups $[L]$ and $[M]$ are coresidual with the common residual group $[N]$. If further $[L']=[M']$ with N' as a common residual, then we have

$$[L+L']=[M+M'];$$

for, we have $[L+N]=0$ and $[L'+N']=0$

$\therefore [L+L'+N+N']=0$. Similarly, $[M+M'+N+N']=0$.

Hence $[L+L']$ and $[M+M']$ are coresidual with the common residual group $[N+N']$. Further, we have

$$[L+M']=[L'+M], \qquad \text{or,} \quad [M-M']=[L=L'].$$

Again, the equation $[L+M-L]=[N]$ signifies the same fact as $[L+M]=[L+N]$, which is the same thing as $[M]=[N]$.

From all these it is seen that we can apply the rules of addition and subtraction to the residual equations.

36. (*a*) *If two groups* $[L]$ *and* $[M]$ *be coresidual, any system* $[N]$ *which is residual of one will be a residual of the other.*

For, we have $[M-L]=0$ and $[L+N]=0$

$$\therefore \ [M+N]=0.$$

36(*b*) *Two groups which are coresidual to the same group are coresidual to each other.*

Let the groups $[L]$ and $[N]$ have a common residual $[L']$, and let $[M]$ and $[N]$ have a common residual group $[M']$, so that $[L]$ and $[M]$ are coresidual to the same group $[N]$. Then the groups $[L]$ and $[M]$ are coresidual to each other.

We have (1) $[L+L']=0$, (2) $[N+L']=0$,

(3) $[M+M']=0$, (4) $[N+M']=0$.

From (2) and (4), we have $[L'-M']=0$,

and therefore from (3), $[L'+M]=0$, ... (5)

i.e. $[L']$ is residual to $[M]$.

Similarly, $[M'+L]=0$ ... ... (6)

i.e. $[M']$ is residual to $[L]$.

Now, from (1) and (5), [L] and [M] are coresidual with the common residual [L′]. Similarly, from (3) and (6), [L] and [M] are coresidual with the common residual [M′].

Cor. If two coresidual groups consist each of a single point, the two points coincide.

Ex. 1. If of the $3(m+n)$ intersections of a curve of the $(m+n)$th order with a cubic, $3n$ lie on a curve of the nth order, the remaining $3m$ lie on a curve of the mth order.

Let the two groups of $3n$ and $3m$ points be denoted by [L] and [M], respectively.

Then $[L+M]=0$ and $[L]=0$ $\therefore$ $[M]=0$; *i.e.*, the $3m$ points in [M] are the complete intersections of a cubic with a curve of order k, such that $3k=3m$, $\therefore$ $k=m$, *i.e.*, the curve is of order m.

Ex. 2. The tangents at n collinear points of an n-*ic* meet the curve again in $n(n-2)$ points lying on an $(n-2)$-*ic*.

Ex. 3. If six of the intersections of a cubic and a quartic lie on a conic, the other six lie on another conic.

37. Brill-Nöther's Residual Theorem :*

If two groups of points [L] and [L′] are coresidual with a common residual [M], and if [M′] is any other residual of [L], then [M′] is also residual to [L′]; [L] and [L′] are consequently coresidual with respect to [M′], and to each residual of any of the two groups [L] and [L′].

Consider the four groups of points [L], [L′], [M], [M′] on a basis curve.

Now, if $[L+M]=0$, $[L'+M]=0$, and $[L+M']=0$,

then we have from the first and third $[M]=[M']$. Substituting this in the second, we obtain $[L'+M']=0$;

i.e. [M′] is a residual of [L′];

$\therefore$ [L] and [L′] are coresidual with the common residual group [M′].

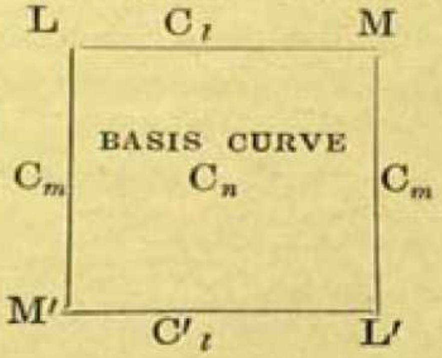
* Brill-Nöther, Über die algebraischen Funktionen und ihre Anwendung in der Geometrie, Math. Ann., Vol. VII (1874), pp. 271-276.

This theorem may be generalised as follows:—

If the groups $[L_1], [L_2], [L_3], \ldots [L_r]$ on a basis curve are coresidual with a common residual group $[M]$, then they are also coresidual with respect to any other residual group of any one of them.

38. Extension of the Residual Theorem:

The direct proof of Brill-Nöther's theorem depends entirely on a similar identity as used in the proof of § 32, which is evidently a particular case of the theorem. We can, however, represent this residual theorem by means of the adjoining diagram.



By introducing a new auxiliary curve $C'_{n'}$, we may express the identity in the form:

$$C_m C_{m'} \equiv C_n C'_{n'} + C_l\, C'_{l'} \qquad \ldots \quad (1)$$

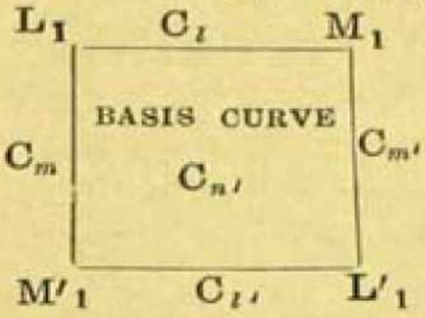
The orders of these curves must then satisfy the conditions

$$m + m' = n + n' = l + l' \qquad \ldots \quad (2)$$

If the identity (1) holds, the theorem is proved. This is the case in particular, if the curves C_m, $C'_{n'}$, C_l, $C'_{l'}$ are "Adjoints" to C_n, *i.e.*, if they pass $(i-1)$ times through an i-ple point of C_n. In the theory of groups of points on a curve, "adjoint" curves play an important part. We, however, postpone discussion till a later chapter.

In the identity (1), the curve $C'_{n'}$ may as well be taken as the basis curve and it may be treated exactly as the curve C_n.

Since C_m, C_l, $C_{m'}$ and $C_{l'}$ intersect mutually in the other four groups of points L_1, M_1, L'_1, M'_1, which by (1) lie on $C'_{n'}$, they may be represented by means of the accompanying diagram.

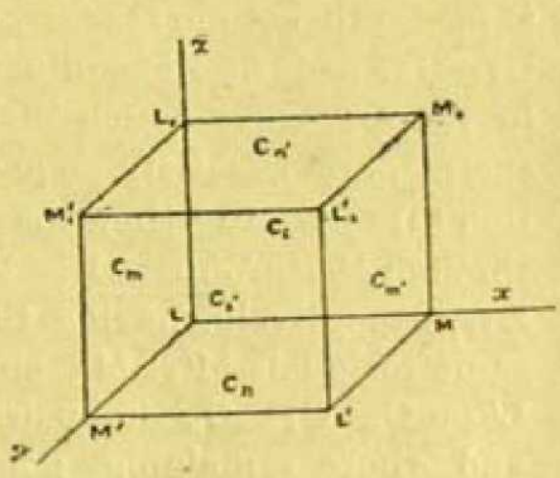


Therefore, we may write Nöther's identity in the following symmetrical form:

$$C_m C_m' + C_l C_l' + C_n C_n' = 0. \qquad \dots \quad (3)$$

Thus, supposing we deal with singular curves, we recognise the completely identical nature of all the six curves on which lie the four groups of points satisfying Brill-Nöther's theorem. From this extension a number of other theorems on configurations of these points can be deduced.*

We can represent the identity by means of a cube as well, shown in the diagram, if the squares in the preceding diagrams are taken as its base and top. The eight groups of points will be at the corners, and the six curves will be taken as corresponding to the six faces.



If we take LM, LM′ and LL_1 as the axes of x, y and z, and the cube of unit edge, the faces $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$ will represent the curves C_m, $C_{m'}$, C_l, $C_{l'}$, C_n, $C_{n'}$ respectively. Each curve passes through the

* E. Study in Marburg, "Ueber Schnittpunktfiguren ebener algebraicher Kurven," Math. Ann., Bd. 36 (1890), pp. 216-229.

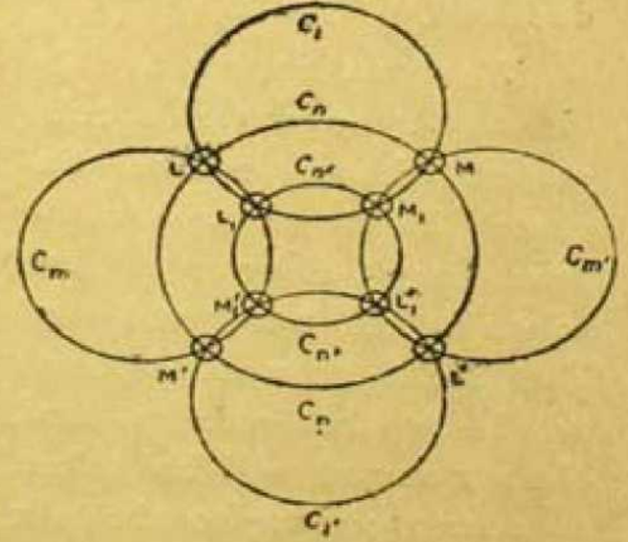
points lying at the four corners of the corresponding face. The complete intersection of any two curves consists of the points at the ends of the common edge of the adjacent faces representing the curves. Thus, the curve C_l passes through the points L, M, L_1, M_1; and the curves C_l and C_m intersect in the points L, L_1, and so on.

39. We may illustrate the above facts very clearly by means of the following.

Each curve is replaced by a circle and each group of points is denoted by a single corresponding point.

Instead of starting with one curve as a basis curve, we may start with one group of points, and state the theorem in the following form:

If three algebraic curves C_m, C_l, C_n pass through the same group of points L and intersect by pairs in the groups, L_1, M′ and M, then three other curves $C_{m'}$, $C_{l'}$, $C_{n'}$ can be determined in various ways, such that their orders satisfy equation (2) only, and they pass through the groups M, M′, L_1, respectively, having the same residuals M'_1, M_1 and L′ on C_m, C_l, C_n in pairs, and whose remaining point of intersection together form a single group L'_1.



Ex. 1. Three conics through the same two points I and J intersect in pairs in three other pairs of points. The three lines joining them are concurrent.

Ex. 2. A conic meets a quartic in eight points. If another conic touches the quartic at four of these points, a third conic touches the quartic at the remaining four points.

Ex. 3. A conic passes through four fixed points of a cubic. Show that the line joining the other two intersections of the conic and cubic meets the cubic again at a fixed point.

Ex. 4. Through a group of points P on an *n-ic*, a curve is drawn to intersect the *n-ic* again in the group P_1, through P_1 is drawn a curve to intersect the *n-ic* again in the group P_2, and so on. Show that if the final group consists of a single point, the same point will be obtained whatever be the curves used.

Ex. 5. If three cubics pass through seven common points, the lines joining the remaining intersections of the cubics taken in pairs form a triangle whose vertices lie one on each of three given cubics.

Ex. 6. If a conic meets a quartic in eight points and a conic touches the quartic at four of the points, a conic touches the quartic at the other four.

CHAPTER III

Singular Points on Curves.

40. In this chapter we shall study singular points and lines of curves, and with this end in view, we shall first study the nature of intersections of a radius vector drawn through the origin to intersect the curve.

We may prove the following theorem:

If the origin of co-ordinates lies on the curve, the constant term in the general equation of a curve vanishes, and the linear terms equated to zero give the equation of the tangent at the origin.

The general equation of a curve of the nth degree (§ 20), when transformed to polar co-ordinates by the substitution $x=r\cos\theta$, $y=r\sin\theta$, may be written as—

$$\begin{aligned}&a\\&+r(b\cos\theta+c\sin\theta)\\&+r^2(d\cos^2\theta+e\cos\theta\sin\theta+f\sin^2\theta)\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&+r^n(p\cos^n\theta+q\cos^{n-1}\theta\sin\theta+\ldots\ldots+s\sin^n\theta)=0 \quad \ldots \quad (1)\end{aligned}$$

This equation determines the distances from the origin of the points in which the radius vector meets the curve.

When the origin lies on the curve, one root of equation (1) must be zero, which requires that $a=0$, whatever be the value of θ.

If, however, θ be so determined that

$$b\cos\theta+c\sin\theta=0 \quad \ldots \quad (2)$$

two values of r will be zero, and the line drawn through the origin in the direction given by (2) will meet the

curve in two coincident points at the origin and will be a tangent at the point.

The equation of this tangent is therefore

$$r(b \cos \theta + c \sin \theta) = 0, \ \textit{i.e.}, \ bx + cy = 0 \quad \dots \quad (3)$$

Note. If $b=0$, the axis of x is a tangent, and if $c=0$, then axis of y is a tangent.

41. If, however, in the general equation, $a=b=c=0$, then, whatever be the value of θ, the co-efficient of r will always be zero, and consequently, two values of r will always be zero, and every line drawn through the origin meets the curve in two coincident points at the origin. The origin, in this case, called a *double point* on the curve.

Thus, every line drawn through a double point meets the curve in two coincident points there. This point, in fact, is one where the curve cuts itself once, *i.e.*, two branches of a curve cut each other at a *double* point.

If we determine θ, so as to satisfy the equation

$$d \cos^2 \theta + e \cos \theta \sin \theta + f \sin^2 \theta = 0$$

then three values of r will be zero. The equation giving the values of θ is a quadratic, and therefore gives two values of $\tan \theta$. Thus we see that although every line drawn through a double point meets the curve in two coincident points, there are two particular lines corresponding to these two values of $\tan \theta$, which meet the curve in *three* coincident points, or have a contact of the *second* order with the curve at the origin. These two lines are the tangents at the double point and their equation is therefore

$$r^2(d \cos^2 \theta + e \cos \theta \sin \theta + f \sin^2 \theta) = 0$$

$$\textit{i.e.}, \ dx^2 + exy + fy^2 = 0 \quad \dots \quad (4)$$

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Hence we obtain the theorem :—

If the origin be a double point on the curve, the terms of the lowest degree form a quadratic, which equated to zero gives the equation of the two tangents at the double point.

DEFINITION. Curves possessing double points are called *autotomic* (self-cutting), and curves not possessing these singularities are called *anautotomic* (or non-singular).

Ex. Show that the point $(0, -1)$ is a double point on the curve $x^3 + y^3 + 3(y+1)(y-2x) + 1 = 0$.

42. Points of Inflexion :

If, however, in the equation of § 40, the value of θ determined by

$$b \cos \theta + c \sin \theta = 0$$

makes the co-efficient of r^2 also vanish, *i.e.*, if the same value of θ satisfies both the equations—

$$b \cos \theta + c \sin \theta = 0$$

and $d \cos^2\theta + e \cos\theta \sin\theta + f \sin^2\theta = 0$, the radius vector meets the curve at three points coinciding with the origin. The origin is called a *point of inflexion* or simply *an inflexion*, and the tangent is called the *inflexional tangent*.

We may therefore define a *point of inflexion* on a curve as a point, which is not a double point, where the tangent has a contact of the second order with the curve. In fact, if three consecutive points on a curve are collinear, there is an inflexion. The tangent at such a point is called a *stationary* tangent or an *inflexional* tangent.

In this case it is evident that $b \cos\theta + c \sin\theta$ is a factor of $d \cos^2\theta + e \sin\theta \cos\theta + f \sin^2\theta$, or, what is the same thing, $bx + cy$ is a factor of $dx^2 + exy + fy^2$

or, in other words, the terms of the second degree contain the linear terms as a factor. Hence the equation of a curve, having the origin for a point of inflexion, may be written as

$$F \equiv u_1 + u_1 v_1 + u_3 + u_4 + \dots\dots + u_n = 0.$$

We may therefore enunciate the following theorem :—

If the terms of the first degree are a factor of the terms of the second degree, the tangent meets the curve in three points coinciding with origin and is called a tangent of three pointic contact.

It is to be noticed that the curve in this case crosses the tangent.

43. Point of Undulation :

If, again, in the same equation, the co-efficient of r^3 vanishes for the same value of θ, the tangent meets the curve in four points coinciding with the origin, which is then called a *point of undulation*, and the tangent has a four-pointic contact.

These results are generalised in the following form : –

If the terms of the first degree are a factor of the terms of the 2nd, 3rd......$(r-1)$th degrees, the tangent has r-pointic contact with the curve at the origin. The tangent crosses the curve or not, according as r is odd or even.

Ex. 1. Investigate the nature of the origin on the curves

(i) $y(x+y+1)+x^3=0$ (ii) $y\{(x-y)^2-2\}=x-y$

(iii) $x+x^2+xy+x^3+2xy^2-x^2y+y^4=0.$

Ex. 2. Show that the origin is an undulation on the curve $x(1+x+x^2+x^3)+y^4=xy(1-y)$ with the axis of y as the tangent.

44. Multiple Points:

If $a=b=c=d=e=f=0$ in the general equation, then whatever be the value of θ, the co-efficients of r and r^2 in (1) will always be zero, and the equation gives three zero values of r. In this case every line drawn through the origin meets the curve in three coinciding points at the origin, which is now called a *triple point*. There are, of course, three of these lines, whose directions are determined by putting the co-efficient of r^3 equal to zero, which meet the curve in *four* points coincident with the origin; and they are the tangents at the triple point. The third degree terms equated to zero give the equation of these tangents.

In general, if the lowest terms in the equation of a curve be of degree k, the origin is a *multiple point of order* k on the curve. Every line drawn through this point has a contact of the $(k-1)$th order with the curve, but there are k of these lines which have a contact of the kth order, and are called the *tangents at the multiple point*.

The equation of the curve in this case can be put into the form—

$$u_k + u_{k+1} + \dots\dots + u_n = 0$$

and $u_k = 0$ gives the k tangents at the origin.

In case of homogeneous equations, the degree in the variables of the co-efficient of the highest power terms determines the multiple point of that order at the corresponding vertex.

Ex. 1. Examine the nature of the origin on the curve

$$(x^2+y^2)^2=(3x^2y-y^3).$$

Ex. 2. Show that the point $(1,-2)$ is a triple point on the curve

$$x^4-4x^3-2x^2y+4xy^2+y^3+2x^2+6y^2+4x+10y+5=0.$$

Ex. 3. Show that the curve $(x^2+y^2)^5=3x^2y^2(x^2-y^2)^2$ has a multiple point of order eight at the origin.

45. Investigation in Homogeneous Co-ordinates:

The general equation of the nth degree in trilinear co-ordinates may be written as—

$$u_0x^n + u_1x^{n-1} + u_2x^{n-2} + \dots\dots\dots\dots + u_n = 0 \qquad \dots \quad (1)$$

where u_r is a binary quantic ($r-ic$) in y and z.

If the curve passes through the vertex A of the fundamental triangle, the equation (1) must be satisfied by $y=z=0$, which requires that $u_0=0$, or the co-efficient of the highest power of x is zero.

Hence, *if a curve passes through the angular points of the fundamental triangle, the co-efficients of the nth powers of x, y, z are absent from the equation.*

If we wish to determine the points where the line $u_1=0$ intersects the curve, we eliminate z between $u_1=0$ and the equation (1). The resultant equation will contain y^2 as a factor, which shows that $u_1=0$ touches the curve where the side CA or $y=0$ cuts it.

Hence, *when the curve passes through the angular points of the fundamental triangle, the co-efficients of the $(n-1)$th powers of x, y, z equated to zero give the tangents at these points respectively.*

If the point A is a double point on the curve, $u_0=0$, $u_1=0$, and $u_2=0$ is the equation of the two tangents at the double point.

Hence, *if the angular points of the fundamental triangle are double points on the curve, the co-efficients of the $(n-2)$th powers of x, y, z give the tangents at these points.*

In general, if the angular points are multiple points of the kth order, the co-efficients of all powers up to $(n-k-1)$th of x, y, z are absent, and the co-efficients of the $(n-k)$th powers equated to zero give the tangents at these multiple points.

If the point A is a point of inflexion on the curve, $u_0=0$, and u_1 is a factor of u_2, and $u_1=0$ is the inflexional

tangent. If u_1 is also a factor of u_3, A is a point of undulation, and so on. In general, if the highest power of x which occurs in the homogeneous equation of a curve of order n is x^{n-k}, the vertex A is then a multiple point of order k and the co-efficient of x^{n-k} equated to zero gives the tangents at the multiple point. If, however, the co-efficients of x^{n-k}, x^{n-k-1}......x^{n-k-r} have a common factor, the corresponding tangent has $(r+2)$-pointic contact with the branch it touches.

Ex. 1. The sides CA and CB of the triangle of reference has r-pointic contact at A and B with an *n-ic*. Show that its equation takes the form

$$xyu_{n-2} = z\ u_{n-}$$

where u_k is homogeneous of degree k in x, y, z.

Ex. 2. An *n-ic* has three tangents having n-pointic contact. Taking these tangents as the sides of the triangle of reference, its equation can be written as

$$xyz\ u_{n-3} + (ax \pm by)^n + (by + cz)^n + (cz + ax)^n = a^n x^n + b^n y^n + c^n z^n$$

the sign $+$ of the ambiguity being taken, if n is odd, and either sign, if n is even.

Ex. 3. Prove that the points of contact in *Ex.* 2 are collinear, if n is odd.

Ex. 4. A sextic has three-pointic contact with each of $x=0$ and $y=0$ at two distinct points. Show that its equation is of the form

$$xyu_4 = u_2^3.$$

Ex. 5. Show that the point (α, β) is a node on the curve

$$\alpha^2\phi + \alpha\beta\psi + \beta^2\chi = 0$$

where α, β are linear functions and ϕ, ψ, χ any functions of the co-ordinates.

Ex. 6. Show that (α, β) is a point of inflexion on the curve $\alpha\phi + \beta^3\psi = 0$, with $\alpha = 0$ as the inflexional tangent, where α, β are linear and ϕ, ψ any functions of co-ordinates.

Ex. 7. Given the equation of a curve in the form

$$Au^2 + 2Buv + Cv^2 = 0$$

where u, v, A, B, C are any functions of the co-ordinates. Show that the common points of u and v are double points on the curve.

If u and v are linear, show that $A'u^2+2B'uv+C'v^2=0$ represents the tangents at the double point (u, v), where A', B', C' are the values of A, B, C when $u=0$ and $v=0$ are substituted in them.

46. A multiple point of order k on a curve may be regarded as a point through which there pass k *distinct* branches of the curve.

Hence, *a multiple point of order k may be considered as arising from the union of $\frac{1}{2}k(k-1)$ double points.*

Consider the curve as consisting of k distinct branches, which do not all pass through a common point. Each point of intersection of two distinct branches is a double point. Therefore there are $\frac{1}{2}k(k-1)$ double points formed by the mutual crossing of the k branches. But when all the k branches pass through a common point, all these double points coincide at that point, which then becomes a multiple point of order k.

The case of k right lines furnishes a simple illustration. The k lines mutually intersect in $\frac{1}{2}k(k-1)$ points. These become coincident when all the k lines pass through a common point, which is clearly a multiple point of order k.

It should, however, be noted here that there is a limit to the number of double points which can be replaced by a multiple point of higher order. For example, a quintic may have six nodes, and only three of them can be replaced by a triple point, and the other three cannot. For in that case, the line joining the two triple points would meet the quintic in six points, which is absurd.

Generally, if an n-*ic* has an $(n-2)$-ple point, it can have only double points, and that again not more than $(n-2)$ in number.

47. Conditions for a Double Point:

Let $f(x, y)=0$ be the Cartesian equation of a curve of order n, and consider the intersections of the right line

$$\frac{x-x'}{l} = \frac{y-y'}{m} = r$$

drawn through a given point (x', y') with the curve. Now any point on this line has co-ordinates

$$(x'+lr, \quad y'+mr)$$

and if this lies on the curve $f(x, y)=0$, we must have

$$f(x'+lr, \ y'+mr)=0,$$

which, by Taylor's Theorem, becomes—

$$f(x', y') + r\triangle f + \frac{r^2}{2!}\triangle^2 f + \ldots\ldots\ldots\ldots + \frac{r^n}{n!}\triangle^n f = 0 \qquad \text{(A)}$$

$$\text{where} \quad \triangle = l\frac{\partial}{\partial x'} + m\frac{\partial}{\partial y'}.$$

Now, if the point (x', y') lies on the curve, $f(x', y')=0$ and one root of the equation (A) becomes zero.

If, however, $l : m$ be so determined that

$$l\frac{\partial f}{\partial x'} + m\frac{\partial f}{\partial y'} = 0$$

then the co-efficient of r vanishes, and another root of (A) becomes zero; *i.e.*, the line drawn in this direction meets the curve in two coincident points at (x', y'), which is therefore a tangent, and its equation is

$$(x-x')\frac{\partial f}{\partial x'} + (y-y')\frac{\partial f}{\partial y'} = 0$$

which reduces to

$$x\frac{\partial f}{\partial x'} + y\frac{\partial f}{\partial y'} + z\frac{\partial f}{\partial z'} = 0 \qquad \ldots \quad \text{(B)}$$

when the equation is made homogeneous by introducing a third variable $z(=1)$.

If, however, $\dfrac{\partial f}{\partial x'}=0, \quad \dfrac{\partial f}{\partial y'}=0$

then the co-efficient of r vanishes identically, and all lines drawn through the point (x', y') meet the curve in two coincident points, irrespective of their directions. The point (x', y') is therefore a double point on the curve.

Thus, at a double point (x', y') on a curve, we must have

$$\frac{\partial f}{\partial x'}=0 \quad \text{and} \quad \frac{\partial f}{\partial y'}=0 \qquad \ldots \quad (C)$$

The equation of the tangents at the double point is given, as before, by

$$(x-x')^2 \frac{\partial^2 f}{\partial x'^2}+2(x-x')(y-y')\frac{\partial^2 f}{\partial x' \partial y'}$$

$$+(y-y')^2 \frac{\partial^2 f}{\partial y'^2}=0 \qquad \ldots \quad (D)$$

If the equation of the curve be given, in any homogeneous system of co-ordinates, in the form $f(x, y, z)=0$, we may find, in a like manner, that any point (x', y', z') is a double point on the curve, if

$$\frac{\partial f}{\partial x'}=0, \quad \frac{\partial f}{\partial y'}=0, \quad \frac{\partial f}{\partial z'}=0 \qquad \ldots \quad (E)$$

In this case, it is possible to eliminate (x', y', z') between the equations (E), and the result is that the discriminant of the equation $f(x, y, z)=0$ vanishes.

Hence, *the condition that a curve has a double point is that the discriminant of its equation vanishes.* The degree of this discriminant is $3(n-1)^2$ in the co-efficients of the equation.

48. Species of Double Points:

We have seen (§41) that there are two tangents to a curve at a double point. Now, these tangents may be (1) *real* and *distinct*, (2) *coincident*, (3) *both imaginary.* Double points can therefore be divided into *three* different classes corresponding to these three cases:—

Case I. When the two tangents at a double point are both real and distinct, there are two real branches of the curve passing through the point, which is then called a *node* or *crunode*.

Case II. If the two tangents at a double point be real but coincident, the two branches of the curve touch at that point, which is then called a *cusp* or a *spinode*.

Case III. If the tangents at a double point be imaginary, there are no real points on the curve consecutive to the double point, which is then called a *conjugate point*, or an *acnode*.

In fact, a conjugate point is an *isolated* point, whose co-ordinates satisfy the equation of the curve, but the point itself does not appear to lie on it. The existence of such a point is geometrically manifest by showing that there are points no line through which can meet the curve in more than $(n-2)$ points.

49. Investigation of the Species of Double Points:

We have seen (§47) that the equation of the tangents at the double point (x', y') on the curve $f(x, y)=0$, is given by

$$(x-x')^2 \frac{\partial^2 f}{\partial x'^2} + 2(x-x')(y-y') \frac{\partial^2 f}{\partial x' \partial y'}$$

$$+ (y-y')^2 \frac{\partial^2 f}{\partial y'^2} = 0 \quad \cdots \quad (1)$$

If θ be the angle between these two tangents, we have

$$\tan\theta\,^{*}=\frac{2\sqrt{\left(\dfrac{\partial^2 f}{\partial x'\partial y'}\right)^2-\dfrac{\partial^2 f}{\partial x'^2}\dfrac{\partial^2 f}{\partial y'^2}}}{\dfrac{\partial^2 f}{\partial x'^2}+\dfrac{\partial^2 f}{\partial y'^2}}$$

The tangents will therefore be real and distinct, coincident or imaginary, according as

$$\left(\frac{\partial^2 f}{\partial x'\partial y'}\right)^2 \gtreqless \frac{\partial^2 f}{\partial x'^2}\cdot\frac{\partial^2 f}{\partial y'^2} \qquad \ldots \quad (2)$$

Thus, the point (x', y') will be a node, a cusp† or a conjugate point, according as the conditions (2) are satisfied, provided

$$f(x', y')=0, \quad \frac{\partial f}{\partial x'}=0, \quad \frac{\partial f}{\partial y'}=0.$$

The tangents will be mutually at right angles, if

$$\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial y'^2}=0$$

and therefore, $\qquad \dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$

represents a curve which cuts $f(x, y)=0$ in all the double points at which the two tangents are mutually at right angles.

* Salmon, Conics, §74.

† The case of the coincidence of the tangents must be examined by a special method, for it is seen that, in some cases, the curve becomes imaginary in the vicinity of the point, even when the above condition for a cusp is satisfied. The point ought then to be regarded as a conjugate point. But the cusp is a distinct singularity. It occurs as a double point, simply because it satisfies the analytical conditions for such a point. Here we do not purpose to enter into a detailed investigation of the species of cusps, consideration of which is postponed to a subsequent chapter.

In a like manner, we may investigate the species of double points on a curve, where its equation is given in any system of homogeneous co-ordinates.

Thus, the equation of the tangents at any double point may be obtained, in a like manner, as

$$x^2\frac{\partial^2 f}{\partial x'^2}+y^2\frac{\partial^2 f}{\partial y'^2}+z^2\frac{\partial^2 f}{\partial z'^2}+2yz\frac{\partial^2 f}{\partial y'\partial z'}$$
$$+2zx\frac{\partial^2 f}{\partial z'\partial x'}+2xy\frac{\partial^2 f}{\partial x'\partial y'}=0$$

and the double point is a node, a cusp or a conjugate point, according as these lines are real and distinct, coincident or imaginary. The conditions are :—The line-pair will be imaginary, if any one of the functions

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

is positive. It will be real, if no one of these functions is positive, *i.e.*, if everyone is either zero or negative. This is ensured, if any one such function is negative or if two be zero. These may be deduced from the results in Cartesian system by replacing x and y by x/z and y/z respectively.

Ex. 1. Examine the nature of the origin on the curves :

(*i*) $x^3+y^3=3x^2+y^2-2xy$ (*ii*) $x^3y-x^3+y^2=0$

(*iii*) $x(x+y)=y^3-y^4$.

Ex. 2. Find the double points of the curves :

(*i*) $x^4-2y^3-3y^2-2x^2+1=0$. (*ii*) $z^2x=y^2(y-x)$.

(*iii*) $4(x-1)^3+(y-3x+2)^2=0$. (*iv*) $x^3+y^3+3axy=0$.

(*v*) $2(x+y+z)^3-54xyz=0$.

Ex. 3. For what value of k, the curve $x^3+y^3+z^3=k(x+y+z)^3$ has a double point?

Ex. 4. Discuss the nature of the cuspidal tangent of the curve

$$(by-cx)^2=(x-a)^5.$$

Ex. 5. Find the double points on the curve $x^4(x+b)=a^3y^2$.

50. Relation between Co-efficients:

From what has been said above, it follows that the fact that a curve has a node is equivalent to only *one* relation between the co-efficients, namely, the result of eliminating x, y from

$$f(x, y) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \qquad \dots \quad (i)$$

But if a given point is to be a node on the curve, it is equivalent to *three* relations between the co-efficients, namely, the relations given by (i).

Similarly, if a curve has a cusp, it is equivalent to *two* relations obtained by eliminating x, y from (i) and

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} \qquad \dots \quad (ii)$$

But if a *given* point is to be a cusp, it is equivalent to *four* relations between the co-efficients, given by (i) and (ii).

Thus, *a curve of order n with δ nodes and κ cusps can be made to satisfy $\frac{1}{2}n(n+3) - \delta - 2\kappa$ relations; but if the nodes and cusps are to be at assigned points, the number of relations is $\frac{1}{2}n(n+3) - 3\delta - 4\kappa$.*

In general, if a curve has a multiple point of order k, this is equivalent to $\frac{1}{2}k(k+1)$ conditions. If, however, an assumed point is a multiple point of order k on a curve, the co-efficients are connected by $\frac{1}{2}k(k+1) - 2$ relations.

It should be noted that these results are not universally true, and due caution must be taken in their applications.

Ex. 1. Show that one n-*ic* in general can be drawn with a given node and passing through $\frac{1}{2}(n^2 + 3n - 6)$ other given points.

Ex. 2. If a point is to be an inflexion on a curve, that amounts to three conditions (§ 42).

Ex. 3. Show that $\alpha L^2 + \beta LM + \gamma M^2 = 0$, where α, β, γ, L and M are linear, is the equation of a cubic with a given node.

51. Intersection of Curves at Singular Points:

If a curve of the mth degree intersect a curve of the nth degree in a double point on the latter, then the point counts as *two* among the intersections, and consequently they can intersect only in $mn-2$ other points. If the point be a double point on both, the intersection must be counted as *four*. In general, if the point of intersection be a multiple point of order k on one and l on the other, it counts as kl of the intersections. Thus we obtain the theorem :—*

If two curves have a common multiple point with different tangents, the number of their intersections, coincident at that point, is equal to the product of the orders of multiplicity of the point on each of the two curves.

Again, if the two curves have a common tangent at that point, it counts at $kl+1$ intersections, for they have one other point common on the tangent. Thus, if they have r tangents common, the point is to be counted as $kl+r$ intersections. Hence we obtain the theorem :—†

If two curves have common tangents at a multiple point on both, the number of their intersections, coincident at that point, is equal to the product of the orders of multiplicity of the point on each, increased by the sum of the orders of contact of the branches of the curves.

In particular, when two curves intersect at a point, which is a node on both, the point counts as *four* intersections. If further, they have the same nodal tangents, they have two other consecutive points common, and the point counts as *six* intersections.

* Halphen, Bull. de la Soc. de France, Tom. 1, p. 133.

† This proposition is due to Cayley, the proof of which has been upplied by Halphen, Memoire sur les points singuliers des courbes algebriques.

If, however, it be a node on one and a triple point on the other, the point counts as *six* among the intersections of the curves. But if the two nodal tangents are also tangents at the triple point, the curves have two more consecutive points common. Consequently this point counts as *eight* among the intersections.

Ex. 1. If a degenerate n-*ic* has $\frac{1}{2}n(n-1)$ nodes, it consists of n right lines.

Ex. 2. A curve of order n cannot have two multiple points of orders k_1 and k_2, if $k_1+k_2>n$.

Ex. 3. Discuss the nature of intersections of the curves

$$x^3=z(x^2\pm y^2) \quad \text{and} \quad 3xy^2=z(y^2\pm x^2).$$

52. Limit to the Number of Double Points:

We have seen that every line drawn through a double point on a curve intersects the curve in two coincident points. Hence it follows that a curve of the third order cannot have more than *one* double point; for, if it had two, the line joining these two double points would meet the curve in four points, which is impossible. In a like manner, a quartic curve cannot have more than *three* double points; for, if it had four, through these four double points and one other assumed point on the curve, a conic could be described, which would then intersect the quartic in nine points, which is impossible. Thus it is seen that the number of double points on a curve is not infinite, but there is a limit to the number of such points, depending upon the degree of the curve.

THEOREM.: *A non-degenerate curve of the nth degree cannot have more than* $\frac{1}{2}(n-1)(n-2)$ *double points.*

Let the number of double points on an n-*ic* be N. Then, through these N double points, and through

$$\{\tfrac{1}{2}(n-2)(n+1)-\text{N}\}, \quad \text{or,} \quad \text{N}_1-\text{N} \quad \text{(say)}$$

other ordinary points on the curve, we can describe a curve of the $(n-2)$th order, which is completely determined by $N_1 \equiv \frac{1}{2}(n-2)(n+1)$ points.

Now this curve of order $n-2$ intersects the n-*ic* in $n(n-2)$ points, and each double point counts as two among the intersections. Therefore the total number of intersections of the two curves is—

$$2N + (N_1 - N), \qquad \text{or,} \qquad N + N_1,$$

which, therefore, cannot be greater than $n(n-2)$;

$$
\begin{aligned}
&\textit{i.e.,} && N + N_1 \ngtr n(n-2),\\
&\text{or,} && N \ngtr n(n-2) - N_1,\\
&\textit{i.e.,} && \ngtr n(n-2) - \tfrac{1}{2}(n-2)(n+1),\\
&\textit{i.e.,} && \ngtr \tfrac{1}{2}(n-1)(n-2),
\end{aligned}
$$

i.e., the number of double points N cannot be greater than

$$\tfrac{1}{2}(n-1)(n-2).$$

53. The Deficiency of a Curve :

The *deficiency* * (or genus) of a curve is the number by which the actual number of double points on a curve falls short of the maximum number, which a curve of that degree can possess. Thus, if a curve of order n has δ nodes and κ cusps, and p denotes its deficiency, then

$$p = \tfrac{1}{2}(n-1)(n-2) - \delta - \kappa.$$

It is to be noticed, however, that the deficiency of a non-degenerate curve cannot be negative.

* The notion of deficiency of an algebraic function was introduced by Riemann, "Theorie der Abelschen Functionen" (Crelle, Bd. 54, pp. 115-155), and has been applied to the theory of curves by Clebsch, "Ueber die Anwendung der Abelschen Functionen in der Geometrie"—(Crelle, Bd. 63, pp. 189-243), and others.

For, a curve of order $(n-2)$ can be drawn through the $(\delta+\kappa)$ double points, and other

$$\tfrac{1}{2}(n-2)(n+1)-\delta-\kappa, \text{ or, } (n-2)+p$$

ordinary points on the given curve, since an $(n-2)$-*ic* is determined by $\frac{1}{2}(n-2)(n+1)$ points.

The $(n-2)$-*ic* intersects the given n-*ic* *twice* at each double point, and *once* at each of the $(n-2)+p$ ordinary points.

But they can intersect only in $n(n-2)$ points, and consequently, in

$$n(n-2)-2\{\tfrac{1}{2}(n-1)(n-2)-p\}-\{(n-2)+p\}=p$$

other remaining ordinary points.

Hence p cannot be negative for a non-degenerate n-*ic*. If the n-*ic* is degenerate, the $(n-2)$-*ic* might form part of the n-*ic*, and the above statement fails.

This also follows from the fact that the actual number of double points can never exceed the maximum number and consequently the deficiency can never be negative.

If the curve has a multiple point of order k, it is equivalent to $\frac{1}{2}k(k-1)$ double points (nodes) (§ 46), and consequently, the deficiency p is given by

$$p=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}k(k-1)$$

and in general, if the curve has δ nodes, κ cusps and other multiple points of orders k_1, k_2, k_3..... the deficiency is given by

$$p=\tfrac{1}{2}(n-1)(n-2)-\delta-\kappa-\Sigma\tfrac{1}{2}k(k-1)$$

where Σ extends over all the multiple points of the curve. It will be seen that the deficiency in this case also cannot be negative.

54. Unicursal Curve:

THEOREM. *If a curve has its maximum number of double points, the co-ordinates of any point on it can be expressed rationally in terms of a single variable parameter.*

Assuming that there are no other multiple points, the number of double points on the curve is $\frac{1}{2}(n-1)(n-2)$. Through these double points and $(n-3)$ other *assumed* points on the curve (altogether making up

$$\tfrac{1}{2}(n-1)(n-2)+(n-3) \quad \text{or} \quad \tfrac{1}{2}(n-2)(n+1)-1$$

points) a system of curves of the $(n-2)$th degree can be described (§22). The equation of such a system will involve an arbitrary parameter, and can therefore be written as $U=\lambda V$, where U and V are any two particular curves of the system. Now consider the intersections of this curve and the *n-ic*. The abscissae of the points of intersection will be determined by an equation obtained by eliminating y between their equations and the degree of this eliminant in x will be $n(n-2)$, and the parameter λ will enter in the nth degree.

Now the curves intersect *twice* at each double point. Hence the known double points count as $(n-1)(n-2)$ intersections. Consequently, altogether

$$(n-1)(n-2)+n-3, \quad \text{or,} \quad n(n-2)-1$$

intersections are known, and only one other intersection remains unknown. Therefore all the roots of the above equation *except one* are known. Dividing the equation by the factors corresponding to the $n(n-2)-1$ known roots, the remaining factor determines the value of x as a rational algebraic function of the nth degree in λ.

Definition. A curve is said to be *unicursal* or *rational* when the co-ordinates of any point on it can be expressed

rationally and algebraically in terms of a single variable parameter.

It is called rational, because the co-ordinates are expressed rationally in terms of a parameter. It is called unicursal, in view of the fact that the curve can be drawn by a pencil at a stretch, never leaving the plane of the paper except when passing through a conjugate point or passing from one end of an asymptote to the other. The curve in fact consists of a single circuit.*

If the curve has a multiple point of order k, we may replace it by $\frac{1}{2}k(k-1)$ nodes and proceed as usual.

55. The Converse Theorem :

If the co-ordinates of any point on a curve can be expressed rationally in terms of a variable parameter, the curve has its maximum number of double points, or, what is the same thing—*the deficiency of a unicursal curve is zero.*

Let the curve be defined by the parametric equations—

$$\left.\begin{array}{l} x=f_1(t) \\ y=f_2(t) \\ z=f_3(t) \end{array}\right\}$$

The n points in which the line $lx+my+nz=0$ intersects the curve are determined by

$$lf_1(t)+mf_2(t)+nf_3(t)=0 \qquad \dots \quad (1)$$

If two of these intersections coincide, the line becomes a tangent and the equation (1) will have a double root, which requires that

$$lf'_1(t)+mf'_2(t)+nf'_3(t)=0 \qquad \dots \quad (2)$$

* This is also possible for curves of deficiency other than zero.

Eliminating l, m, n, we obtain the equation of the tangent in the following determinant form—

$$\begin{vmatrix} x & y & z \\ f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \end{vmatrix} = 0 \qquad \dots \quad (3)$$

If we regard (x, y, z) as given and t variable, the equation (3) determines the values of t which correspond to the tangents that can be drawn from the point (x, y, z). The degree of this equation in t is therefore equal to the number of such tangents. But the degree of this equation in t is $2n-2$, for the co-efficient of t^{2n-1} is zero. Hence the number of tangents drawn from the point to the curve is only $2(n-1)$. But, as we shall see later, the number of such tangents is $n(n-1)$. Therefore the number of these tangents for a *unicursal* curve is diminished by

$$n(n-1)-2(n-1), \quad i.e., \quad \text{by} \quad (n-1)(n-2).$$

This diminution is due to the coincidence of some of the points of contact, for the line drawn through (x, y, z) and a node satisfies the condition for a tangent. Hence, assuming that there are only nodes * on the curve, we conclude that this diminution is due to the coincidence of the tangents by pairs at the nodes, and consequently, the number of such points is

$$\tfrac{1}{2}(n-1)(n-2)$$

which is the maximum number of double points for a curve of the nth degree, *i.e.*, the deficiency is zero.

* This does not hold for a cuspidal curve. For, the roots include the parameters of the points of contact of tangents and those of cusps as well, since at these latter points, as will be shown later, we have $f'_1/f_1 = f'_2/f_2 = f'_3/f_3$. Hence, if there are κ cusps and m tangents,

$$m+\kappa \geq 2n-2.$$

But m will be found to be $=n(n-1)-2\delta-3\kappa=2n-2-\kappa+2p$.

$\therefore \; m+\kappa=2n-2+2p.$ Since $p=0$, $m+\kappa=2n-2$.

56. A Second Proof :

That a unicursal curve has its deficiency zero can be proved in a rigorous manner as follows :*

Let the curve be defined, as before, by the equations

$$x=f_1(t), \quad y=f_2(t), \quad z=f_3(t) \quad \ldots \quad (1)$$

Any point P will be a double point on the curve, if for two different values t and t' of the parameter, the same values of the co-ordinates are obtained. Consequently, for a double point, we must have—

$$f_1(t)=f_1(t'), \quad f_2(t)=f_2(t'), \quad f_3(t)=f_3(t'), \quad \text{when } t \neq t'.$$

Therefore to determine the parameters of the double points, we must determine the solutions of the system of equations :

$$\frac{f_1(t)}{f_3(t)}=\frac{f_1(t')}{f_3(t')}, \qquad \frac{f_2(t)}{f_3(t)}=\frac{f_2(t')}{f_3(t')}, \quad \text{when } t \neq t'.$$

$$\text{Let} \quad \left.\begin{aligned} \frac{f_2(t)f_3(t')-f_2(t')f_3(t)}{t-t'} &\equiv \phi_1(t, t') \\ \frac{f_3(t)f_1(t')-f_3(t')f_1(t)}{t-t'} &= \phi_2(t, t') \\ \frac{f_1(t)f_2(t')-f_1(t')f_2(t)}{t-t'} &= \phi_3(t, t') \end{aligned}\right\} \quad \ldots \quad (2)$$

where ϕ_1, ϕ_2, ϕ_3 are symmetric and homogeneous functions of order $(n-1)$ in each parameter t and t'. The parameter of a double point must satisfy the equations

$$\phi_1=0, \ \phi_2=0 \text{ and } \phi_3=0, \text{ when } t \neq t'.$$

Hence the common roots of these equations will give the parameters of the double points.

* This proof is given by A. Clebsch zu Giessen, Crelle's Journal, Bd. 64 (1865), pp. 47-48.

Identically we have—

$$\left.\begin{aligned}\phi_1.f_1(t)+\phi_2.f_2(t)+\phi_3.f_3(t)=0\\ \phi_1.f_1(t')+\phi_2.f_2(t')+\phi_3.f_3(t')=0\end{aligned}\right\}\quad\ldots\quad(3)$$

If we eliminate t' between ϕ_1 and ϕ_2, we shall obtain an equation of degree $2(n-1)^2$ in t.* It is clear then that all those roots of this equation, which also satisfy $\phi_3=0$, will give the double points. If we substitute the roots of this equation and the corresponding values of t' in (3), then either $\phi_3=0$, or $f_3(t)=0$ and $f_3(t')=0$, when $t\neq t'$.

Therefore, for double points we have to reject those values which simultaneously make

$$f_3(t)=0,\quad f_3(t')=0,\qquad \text{when } t\neq t'.$$

The number of such values is $n(n-1)$; for, from the equation
$$f_3(t)-f_3(t')=0,$$
after removing the factor $t-t'$, we obtain an equation of order $(n-1)$ in each of the parameters t and t'. Consequently, for each value of t', there are $(n-1)$ values of t, which satisfy the equation. But there are n values of t' which make $f_3(t')=0$.

Hence, there are $n(n-1)$ values of t which make both

$$f_3(t)=0,\quad f_3(t')=0,\qquad \text{when } t\neq t'.$$

Thus, after rejecting these $n(n-1)$ values of t, the remaining $2(n-1)^2-n(n-1)$, or, $(n-1)(n-2)$ roots give the double points, and the number of such points is therefore $\frac{1}{2}(n-1)(n-2)$, *i.e.*, the curve has its maximum number of double points, and consequently its deficiency is zero.

* The resultant contains the co-efficients of each in the degree of the other, and t occurs in degree $n-1$ in each. (Burnside and Panton; Theory of Equation, Vol. II, § 152.)

57. From what has been said above, it follows that all curves are not, in general, unicursal. The condition, both *necessary* and *sufficient*, that a given curve may be unicursal is that it has its maximum number of double points, *i.e.*, its deficiency is zero.

If f_1, f_2, f_3 be three rational and algebraic functions of a parameter t, and

$$x : y : z = f_1 : f_2 : f_3$$

the eliminant of t from these equations gives the equation of the curve in the implicit form. If f_1, f_2, f_3 are functions of the nth degree in t, we may eliminate the parameter by the dialytic method. The result is given in the form of a determinant, in which the variables enter only in the nth degree. Thus the curve is also one of the nth degree.

Ex. **1.** Any point on an ellipse can be expressed as

$$x = a \cos \theta, \quad y = b \sin \theta.$$

The elimination of θ gives the equation of the locus in the form

$$x^2/a^2 + y^2/b^2 = 1.$$

Ex. 2. Show that the conic $ax^2 + 2hxy + by^2 + 2fy + 2gx = 0$ is unicursal.

The co-ordinates of any point P are given by the formulæ—

$$x = -2\frac{g + ft}{a + 2ht + bt^2}, \qquad y = -2\frac{gt + ft^2}{a + 2ht + bt^2}.$$

Ex. 3. Show that the curve $x^3 + y^3 = 3axy$ is unicursal (Folium of Descartes).

The origin is a node. Take a line $y = tx$, which intersects the curve in two points at the origin. To determine the third point of intersection, we put $y = tx$ in the equation. Thus

$$x^3(1 + t^3) = 3atx^2.$$

The factor x^2 corresponds to the double point, and the remaining point is given by $x(1+t^3)=3at$

$$i.e., \; x=\frac{3at}{1+t^3}. \qquad \therefore \; y=\frac{3at^2}{1+t^3}.$$

In the homogeneous form, we may write

$$x : y : z = 3at : 3at^2 : 1+t^2.$$

Ex. 4. Show that the trinodal quartic

$$a/x^2+b/y^2+c/z^2+2f/yz+2g/zx+2h/xy=0$$

is unicursal.

Ex. 5. Express rationally in terms of a parameter the co-ordinates of any point on the curves

$$(i) \; r=a(1+\cos\theta) \qquad (ii) \; r^2=a^2\cos 2\theta.$$

Ex. 6. Show that the co-ordinates of any point on the cissoid

$$(x^2+y^2)x=ay^2$$

may be expressed as $\quad x=\dfrac{a}{1+\theta^2}, \qquad y=\dfrac{a}{\theta(1+\theta^2)}.$

58. Complex Singularities:

At a node on a curve, one or both the nodal tangents may be stationary tangents.

If in the equation (1) of § 40, the co-efficient of r^2 and r^3 have one factor common—or what is the same thing—if the second and the third degree terms in the equation of the curve have one linear factor common, the corresponding nodal tangent has three-point contact with one branch of the curve, and thus becomes a stationary tangent. The node is called a *flecnode* on the curve, which may be regarded as arising from the union of a node and a point of inflexion.

Similarly, if the third degree terms contain the second degree terms as factor, both the nodal tangents are

stationary tangents, and the origin is called a *biflecnode*, which may be regarded as arising from the union of a node with two inflexions.

Thus, the equation of a curve having a flecnode and a biflecnode at the origin may respectively be written as—

$$0=(ax+by)\ (lx+my)+ (ax+by)(dx^2+exy+fy^2)+u_4+\ldots\ldots$$

and $$0=(ax^2+2hxy+by^2)+ (ax^2+2hxy+by^2)(lx+my)+u_4+\ldots\ldots$$

There are other kinds of singular points arising from the union of two or more singularities described before, and these must be investigated by special methods. We shall have occasion to discuss the nature of some of them in a subsequent chapter, and specially, when dealing with quartic curves.

59. Singular Points at Infinity :

It often happens that a curve possesses singular points at infinity. We shall now explain a method by which such singular points can be determined.

Let ABC be the fundamental triangle, and let any line A′B′ whose equation is

$$z'\equiv lx+my+nz=0$$

intersect CA and CB in A′ and B′ respectively. Now the equation of a curve having a singular point at A′ or B′ can be written down as usual. If now A′B′ is supposed to move off to infinity, it will become the line at infinity, and its equation will then become

$$\mathrm{I}\equiv ax+by+cz=0.$$

Therefore, the equation of a curve having a singular point at infinity on CA or CB is obtained by writing down the equation of the curve having a corresponding singular point at A′ or B′, and then changing z' into I or $ax+by+cz$.

60. To find the equation of a curve having a double point at infinity on the line CA.

The general equation of a curve having a double point (node) at A may be written (§ 45) as

$$u_2x^{n-2}+u_3x^{n-3}+\ldots\ldots+u_n=0,$$

where u_r is a binary quantic (r-*ic*) in y and z. Now, if we change z into I in this equation, the equation of a curve having a double point at infinity on the line CA will be

$$u'_2x^{n-2}+u'_3x^{n-2}+\ldots\ldots\ldots+u'_n=0,$$

where u'_r is a binary quantic (r-*ic*) in I and y, so that

$$u'\equiv f(ax+by+cz,\ y).$$

Again, if we wish to obtain the Cartesian equation of a curve having a double point at infinity on the axis of x, we have only to suppose that the angle at C is a right angle, and then put $z=\text{I}=a$ constant. The equation becomes

$$u_2x^{n-2}+u_3x^{n-3}+\ldots\ldots\ldots+u_n=0$$

where u_2 is of the form $\quad ay^2+b\text{I}y+c\text{I}^2$

and u_n is a polynomial of the nth degree in y. If $a=0$, the line at infinity is a tangent. If $a=b=0$, the double point is a cusp, the line at infinity being the cuspidal tangent.

Ex. 1. Show that the curves

$$(i)\ (a^2-x^2)y=a^3, \qquad (ii)\ x^3+a^3=3axy$$

have nodes at infinity.

Ex. 2. Find the singular points on the curves :

$$(i)\ a^2y=x^3, \qquad (ii)\ ay^2=x^3.$$

Ex. 3. Find the inflexions on the curve

$$x^3+y^3=3axy.$$

61. Multiple Tangents :

Singular points on curves may be divided into two classes—(1) multiple points, (2) points of contact of multiple tangents. We have already discussed the nature of multiple points. We shall now proceed to study the nature of multiple tangents, *i.e.*, the lines which touch the curve in two or more points, or which have a contact of the second or higher order with the curve. For simplicity, we shall examine the conditions under which the axis of x may be a multiple tangent to a curve.

Consider the points where the axis of x $(y=0)$ intersects the curve defined by the general equation. If we put $y=0$ in the equation—

$$a+bx+cy+dx^2+exy+fy^2+\ldots+px^n=0,$$

we obtain, for determining the abscissæ of the points of intersection, the equation

$$a+bx+dx^2+gx^3+\ldots+px^n=0.$$

If $a_1, a_2, a_3, \ldots a_n$ be the roots of this equation, it may be written as

$$p(x-a_1)(x-a_2)(x-a_3)\ \ldots\ (x-a_n)=0$$

and $a_1, a_2, a_3 \ldots a_n$ are the abscissæ of the n points of intersection.

Now, if $a_1=a_2$, two of these intersections coincide, and the axis of x is a tangent to the curve at the point $(x=a_1,\ y=0)$, *i.e.*, when two of the roots are equal, the axis of x is a tangent. If a_1 is imaginary, there will be another pair of coincident imaginary roots, namely $a_3=a_4$, and the axis of x will be a *double tangent* or *bitangent*, touching the curve at *two* imaginary points.

When the equation has two pairs of real and equal roots, the axis of x is a *double tangent* or *bitangent*, touching the curve at two real but distinct points. It

is evident that a curve of order lower than the fourth cannot have any bitangent.

If three roots of this equation become equal, for instance, $a_1 = a_2 = a_3$, then the axis of x meets the curve in three consecutive points. In this case it is called a *stationary tangent*; for, when three consecutive points lie on the tangent, we may consider that the tangent joining the first two consecutive points coincides with the consecutive tangent, *i.e.*, the tangent through the last two points. The point of contact of a stationary tangent is called *a point of inflexion* (§ 42).

In a like manner, it can be shown that the axis of x touches the curve at two or more points, according as the above equation has two or more pairs of equal roots. Again, the line may have a contact of the third or higher order, according as the equation has a root repeated four or more times. These singularities can occur at more than one point.

Ex. 1. The side BC of the triangle of reference ABC is a bitangent to a quartic, B and C being the points of contact. Show that the equation of the quartic is $zu_3 = x^2y^2$.

Ex. 2. A line is drawn through each of the points of contact of a bitangent of a quartic. Show that a cubic touches the quartic at the six points in which these two lines meet the quartic again.

Ex. 3. Show that $\alpha = 0$ is a bitangent at the points $(\alpha\beta)$ and $(\alpha\gamma)$ on the curve $\alpha\phi + \beta^2\gamma^2\psi = 0$, where α, β, γ are linear, and ϕ, ψ any functions of the co-ordinates.

62. Reciprocal Singularities:

We have seen that a curve may be regarded as the locus of points or envelope of lines. In order to discuss properties of multiple tangents, tangential co-ordinates may conveniently be used. In the point-theory, at a double point two different points of the curve coincide. In the line-theory we may have two different tangents to

the curve coinciding into a bitangent. Thus to a double point (crunode or acnode) there corresponds a bitangent with real or imaginary contact. At a cusp, the tracing point first becomes stationary and then reverses the sense of its motion. So also at a point of inflexion, the enveloping line first becomes stationary and then reverses the sense of its motion. Hence we see that to a cusp there corresponds a stationary tangent, and these are distinct singularities. Thus the singularities correspond as follows :—

To a node or a conjugate point (with real or imaginary tangents) corresponds a bitangent with real or imaginary points of contact.

To a cusp with the cuspidal tangent, there correspond a stationary tangent and the inflexion respectively.

In the same way, to a triple, quadruple, etc., point with distinct tangents corresponds a tangent respectively with three, four, etc., distinct points of contact.

In particular, to a triple point with coincident tangents corresponds the tangent at a point of undulation, and so on.

It follows from this, remembering the relation that exists between a curve and its reciprocal, that if we have a curve of order n, having δ nodes, κ cusps and satisfying r other conditions, and if there is only a finite number of such curves, so that

$$\tfrac{1}{2}n(n+3)=\delta+2\kappa+r \quad (\S 50)$$

then the reciprocal is of degree m with τ nodes, ι cusps and satisfying r other conditions; there is only a finite number of them, so that

$$\tfrac{1}{2}m(m+3)=\tau+2\iota+r.$$

Hence $\qquad \frac{1}{2}n(n+3)-\delta-2\kappa=\frac{1}{2}m(m+3)-\tau-2\iota \qquad$ (§ 149)

i.e., a curve and its reciprocal are determined by the same number of conditions.

Ex. 1. If $\phi(\lambda, \mu, \nu)=0$ is the tangential equation of a curve, show that the bitangents are determined by

$$\frac{\partial \phi}{\partial \lambda}=\frac{\partial \phi}{\partial \mu}=\frac{\partial \phi}{\partial \nu}=0.$$

Ex. 2. The three tangents at a triple point are coincident. What is the corresponding reciprocal singularity?

Ex. 3. Show that the curve $y^p z^q=x^{p+q}$ has reciprocal singularities at the vertices B and C of the triangle of reference.

Ex. 4. Find the bitangents and inflexions on the curve $3(x+y)=x$.

Ex. 5. Find the nodes on the curve $(\xi^2+\eta^2)\zeta^2=a^2\xi^2\eta^2$.

Ex. 6. Show that $a^3y=x^4$ has a crunode and $ay^3=x^4$ has an undulation. Locate these singularities (singularities at infinity).

CHAPTER IV

THEORY OF POLES AND POLARS

63. Let $f(x, y, z)=0$ be the equation of a curve of the nth degree in any system of homogeneous co-ordinates (or in Cartesians made homogeneous by introducing requisite powers of $z=1$). We shall examine the points where any line joining two given points intersect the curve by using the method of Joachimsthal.

Let P (x', y', z') be a fixed point and Q (x, y, z) a variable point in the plane. Then the co-ordinates of any point A on PQ, dividing PQ in the ratio $\lambda:\mu$ (where $\lambda+\mu=1$), are

$$\lambda x+\mu x', \quad \lambda y+\mu y', \quad \lambda z+\mu z'.$$

The co-ordinates of points where the line PQ meets the curve are found by substituting these values for x, y, z in the equation of the curve, and then determining the ratio $\lambda:\mu$ from the resulting equation

$$f(\lambda x+\mu x', \lambda y+\mu y', \lambda z+\mu z')=0.$$

This may be expanded in two ways by Taylor's theorem. We have then—

$$\Theta=\lambda^n f+\frac{\lambda^{n-1}\mu}{1!}\triangle f+\frac{\lambda^{n-2}\mu^2}{2!}\triangle^2 f+\ldots\ldots$$

$$+\frac{\lambda^{n-r}\mu^r}{r!}\triangle^r f+\ldots+\frac{\mu^n}{n!}\triangle^n f=0 \quad (1)$$

or

$$\Theta=\mu^n f'+\frac{\mu^{n-1}\lambda}{1!}\triangle' f'+\frac{\mu^{n-2}\lambda^2}{2!}\triangle'^2 f'+\ldots$$

$$+\frac{\mu^{n-r}\lambda^r}{r!}\triangle'^r f'+\ldots\ldots+\frac{\lambda^n}{n!}\triangle'^n f'=0 \quad (2)$$

where

$$f\equiv f\,(x,\,y,\,z), \qquad f'\equiv f(x',\,y',\,z')$$

$$\triangle=\left(\,x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\frac{\partial}{\partial z}\right)$$

$$\triangle'=\left(\,x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}\right).$$

Either of these equations gives the n values of the ratio $\lambda:\mu$. Comparing the co-efficients in the two equations, we obtain the following identities :—

$$f=\frac{1}{n!}\triangle'^{n}\,', \qquad \triangle f=\frac{1}{(n-1)!}\triangle'^{n-1}f'$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\frac{1}{r!}\triangle^r f=\frac{1}{(n-1)!}\triangle'^{n-r}f'$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\frac{1}{(n-2)!}\triangle^{n-2}f=\frac{1}{2!}\triangle'^2 f' \quad \frac{1}{(n-1)!}\triangle^{n-1}f=\triangle' f'$$

$$\frac{1}{n!}\triangle^n f=f'.$$

64. Polar Curves:

The several curves defined by the equations $\triangle f=0$, $\triangle^2 f=0$, etc., are called the "*Polar Curves*" of the point (x', y', z') with regard to $f=0$. The curve $\triangle f=0$ is called the *first* polar of the point (x', y', z') with respect to $f=0$. Similarly, the curves $\triangle^2 f=0$, etc., are called respectively the second, third, etc., polar curves of the point (x', y', z') and the point (x', y', z') is called the *pole*. The equation of the kth polar curve is—

$$\left(x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\frac{\partial}{\partial z}\right)^k f=0$$

or

$$\left(x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}\right)^{n-k} f'=0.$$

the two equations representing the same curve in virtue of the identities of the preceding article. It follows therefore that $\triangle^{n-1}f=0$ or $\triangle' f'=0$, which represents the $(n-1)th$ polar curve, is the polar line, and $\triangle^{n-2}f=0$, or $\triangle'^2 f'=0$, which represents the $(n-2)th$ polar curve, is the polar conic, and so on.

From the mode of forming the equations of polar curves it is clear that successive polars are obtained by performing the operation $\triangle$ successively on f; for instance $\triangle^2 f$ is obtained by operating with $\triangle$ on $\triangle f$, which is the first polar of f. Hence the second polar of (x', y', z') with respect to f is the first polar of the same point with respect to $\triangle f$. Similarly, the third polar of (x', y', z') with respect to f is the first polar of the same point with respect to $\triangle^2 f$ and the second polar with respect to $\triangle f$. In general, since $\triangle^r(\triangle^s f)=\triangle^{r+s}f$, the $(r+s)th$ polar of a point with respect to f is the rth polar of the same point with respect to $\triangle^s f$, *i.e.*, with respect to the sth polar of f. Thus we may state the following general theorem:

The polar curve of any rank of a point is also a polar curve of the same point with respect to all polar curves of a rank lower than its own.

It is to be further observed that

$$\triangle^s(\triangle^r f) = \triangle^r(\triangle^s f) = \triangle^{r+s} f$$

$$\triangle^s f = \triangle^{s-r}(\triangle^r f), \text{ and so on.}$$

This shows that the sth polar of a point with respect to the rth polar of the same point *w.r.t.* the original curve is the rth polar of the point *w.r.t.* the sth polar of the original curve.

Ex. 1. If P lies on the kth polar of Q for an n-*ic*, Q lies on the $(n-k)$th polar of P.

Ex. 2. If P is a node of the $(k-1)$th polar of Q, then Q is a node of the $(n-k)$th polar of P for an n-*ic*.

Ex. 3. Show that the kth polar of the vertices of the triangle of reference are

$$\frac{\partial^k f}{\partial x^k}=0, \qquad \frac{\partial^k f}{\partial y^k}=0, \qquad \frac{\partial^k f}{\partial z^k}=0.$$

Ex. 4. Prove that the polar line of a point on the curve is the tangent at the point.

Ex. 5. Prove that all polar curves of a point on the curve will touch the curve at that point.

65. Investigation of Singular Points:

If $P(x', y', z')$ and $Q(x'', y'', z'')$ be any two points in the plane of a curve $f=0$, the points where the line PQ meets the curve are determined, as in §63, by the equation which we may write in the form:

$$\Theta \equiv \lambda^n f(x', y', z') + \lambda^{n-1}\mu \triangle f(x', y', z')$$
$$+ \tfrac{1}{2}\lambda^{n-2}\mu^2 \triangle^2 f(x', y', z') + \ldots = 0 \quad \ldots \quad (1)$$

where $$\triangle \equiv x'' \frac{\partial}{\partial x'} + y'' \frac{\partial}{\partial y'} + z'' \frac{\partial}{\partial z'}.$$

Now, if one of the points should coincide with (x', y', z'), it is evident that one of the roots of this equation should be $\mu=0$. This requires that $f' \equiv f'(x', y', z')=0$, which is otherwise evident, since, when a point lies on the curve, its co-ordinates satisfy the equation of the curve.

If two of the points where PQ meets the curve should coincide with (x', y', z'), then the above equation should give two values of $\mu=0$, *i.e.*, μ^2 should be a factor of Θ. This requires that both $f'=0$ and $\triangle f'=0$. Now then PQ touches the curve at (x', y', z'), and (x'', y'', z'') is a point on that tangent (or tangents, if more than one). Hence, if (x'', y'', z'') be made current, $\triangle$ becomes

$$x\frac{\partial}{\partial x'} + y\frac{\partial}{\partial y'} + z\frac{\partial}{\partial z'},$$

and the point (x'', y'', z'') lies on the locus

$$x\frac{\partial f'}{\partial x'} + y\frac{\partial f'}{\partial y'} + z\frac{\partial f'}{\partial z'} = 0, \qquad \dots (2)$$

which is the tangent to the curve at the point (x', y', z'). But this is evidently the polar line of the point (x', y', z').

Hence, *the polar line of a point on the curve is the tangent at that point.*

It follows from the preceding article that the polar line of a point with respect to a curve will also be the polar line with respect to each of the polar curves, and since the point (x', y', z') lies on the curve, it is seen that it lies on all the polar curves. Hence the polar curves will have the same tangent at the point (x', y', z').

If, however,

$$\frac{\partial f'}{\partial x'} = \frac{\partial f'}{\partial y'} = \frac{\partial f'}{\partial z'} = 0, \qquad \dots \quad (3)$$

$$\triangle f \equiv x'' \frac{\partial f'}{\partial x'} + y'' \frac{\partial f'}{\partial y'} + z'' \frac{\partial f'}{\partial z'}$$

vanishes identically, whatever x'', y'', z'' may be.

Hence, in this case the line PQ meets the curve always in two points at (x', y', z') for all values of x'', y'', z'', *i.e.*, the point (x', y', z') is a *double point*, and every line drawn through it meets the curve in two coincident points at (x', y', z').

The equations (3) will not, in general, have a common solution, unless a certain condition is satisfied, which of course, is obtained by eliminating x', y', z' between the three equations (3). (§47.)

Again, the equation $\Theta = 0$ will have three roots $\mu = 0$, *i.e.*, PQ will meet the curve in three points at (x', y', z'), if we have

$$f' = 0, \quad \triangle f' = 0, \quad \triangle^2 f' = 0 \qquad \dots \quad (4)$$

These show that in this case the line PQ coincides with the tangent at (x', y', z'), and that every point on it is a point on the polar conic $\triangle^2 f' = 0$ of (x', y', z'), which must therefore reduce to two right lines. Hence, $\triangle^2 f'$ contains $\triangle f'$ as a factor, and the point (x', y', z') is a point of inflexion. (§ 42.)

We have thus indirectly obtained the theorem that *the polar conic of a point of inflexion breaks up into two right lines.*

If, however, $\triangle f'$ and $\triangle^2 f'$ vanish identically, whatever x'', y'', z'' may be, *i.e.*, if the first and second differential co-efficients of f vanish at (x', y', z'), the line PQ always meets the curve in three coincident points at (x', y', z'), which is then a *triple point*. (§ 44.)

Generally, if for any point (x', y', z'), $\triangle^{k-1}f'$ is identically zero, the curve has at this point a multiple point of order k. From the mode of formation of $\triangle f'$'s, it follows that, if $\triangle^{k-1}f' \equiv 0$, then $\triangle^{k-2}f' \equiv 0$, $\triangle^{k-3}f' \equiv 0, \ldots, \triangle f' \equiv 0$.

The equation $\triangle^k f' = 0$ gives then the product of the k tangents at the point in question. For, in the first place, the curve represented by this equation has also at the point a multiple point of order k, since its $(k-1)th$ polar, the point being the pole, is no other than $\triangle^{k-1}f'$, and in consequence, vanishes independently of (x'', y'', z''). But a curve of order k having a k-ple point is necessarily composed of k lines. Hence these lines touch the k different branches of the original curve which pass through the multiple point.

66. Mixed Polars:

The symbolic identities discussed above also hold, if two different poles are taken with respect to the rth and the sth polars respectively.

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any two points in the plane of a curve $f=0$.

Then, if $$\triangle \equiv \left(x_1 \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + z_1 \frac{\partial}{\partial z} \right)$$

and $$\triangle' \equiv \left(x_2 \frac{\partial}{\partial x} + y_2 \frac{\partial}{\partial y} + z_2 \frac{\partial}{\partial z} \right)$$

we have $\triangle^s . \triangle'^r = \triangle'^r . \triangle^s$; whence it follows that the sth polar of P_1 with respect to the rth polar of P_2 for any curve is the rth polar of P_2 with respect to the sth polar of P_1 for the same curve.

The curve obtained by this polar process* is called a "mixed polar" curve of the two points.

* Elliot, Algebra of Quantics, §53.

For a cubic curve, in particular, we obtain the following theorem :

If S_1 and S_2 are the polar conics of two points P_1 and P_2 with respect to a cubic, and if tangents are drawn from P_1 and P_2 to S_2 and S_1 respectively, then the four points of contact will lie on a right line, which is the "mixed polar" line of P_1 and P_2 with respect to the given cubic.

For, the polar conics of P_1 and P_2 are $\triangle f$ and $\triangle' f$ respectively,

where $$\triangle = x_1 \frac{\partial}{\partial x} + \ldots\ldots \text{ and } \triangle' = x_2 \frac{\partial}{\partial x} + \ldots\ldots$$

Now, the polar line of P_1 *w.r.t.* S_2 is the first polar of P_1 *w.r.t.* S_2, *i.e.*, $\triangle.\triangle' f=0$ which is a right line passing through the points of contact of tangents drawn from P_1 to S_2. Since $\triangle.\triangle' f = \triangle.'\triangle f$, the truth of the theorem follows.

Ex. 1. The polar curves of the point (a, a) *w. r. t.* the cubic

$$x^3 + y^3 = a^3 \text{ are :—}$$

Polar line $x + y = a$ Polar conic $x^2 + y^2 = a^2$.

Ex. 2. The equations of the pairs of tangents drawn from the vertices of the triangle of reference to the conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

are $Cy^2 - 2Fyz + Bz^2 = 0$, $Az^2 - 2Gzx + Cx^2 = 0$.

and $Bx^2 - 2Hxy + Ay^2 = 0$ respectively.

where A, B, C, etc., are the co-factors of a, b, c, etc., in the determinant $(a\ b\ c)$.

It follows then that the three pairs of lines—

$ay^2 + byz + cz^2 = 0$, $a'z^2 + b'zx + c'x^2 = 0$ and $a''x^2 + b''xy + c''y^2 = 0$ will touch a conic, if $aa'a'' = cc'c''$.

Ex. 3. Any polar curve of the curve

$$(x/a)^n + (y/b)^n + (z/c)^n = 0$$

is of the same form.

[This curve is called the *triangular-symmetric* curve.]

Ex. 4. If the polar conic of P *w. r. t.* a given *n-ic* has a self-conjugate triangle which is inscribed in a given conic, the locus of P is an $(n-2)$-*ic*.

67. Tangent at any Point:

If two roots of either of the equations (1) and (2) of §63 be equal, the line intersects the curve in two coincident points and is therefore a tangent. Hence the discriminant of the above equation, regarded as a binary quantic in $\lambda:\mu$, equated to zero, will give the equation of the tangents drawn from any point (x', y', z') to the curve. The weight, *i.e.*, degree in the roots of this discriminant is $n(n-1)$,* and therefore $n(n-1)$ tangents can be drawn from any point to the curve.

The calculation of the tangents by this method is not so simple, in general; but the method may conveniently be used in particular cases. Thus for example, in the case of a cubic curve, we obtain the following equations for determining the values of the ratio $\lambda : \mu$—

$$\lambda^3 f+\lambda^2\mu\triangle f+\lambda\mu^2\triangle^2 f+\mu^3 f'=0 \qquad \ldots \quad (1)$$

$$\mu^3 f'+\lambda\mu^2\triangle' f'+\lambda^2\mu\triangle'^2 f'+\lambda^3 f=0 \qquad \ldots \quad (2)$$

Writing, for brevity, $\triangle$ and $\triangle'$ for $\triangle f$ and $\triangle' f'$ respectively, the discriminant of (1) or (2) becomes:

$$\triangle^2[\triangle'^2-4\triangle.f']+f[18f'.\triangle.\triangle'-4\triangle'^3-27f^2f']=0' \ldots \quad (3)$$

This equation is symmetrical in the two sets (x, y, z), (x', y', z'), and of degree 6 in each set. Hence 6 tangents can be drawn from any point (x', y', z') to the cubic.

The form of the equation (3) shows that it represents a locus touching f at the points where the latter meets $\triangle f$. The other points where f meets the locus lies on the curve $(\triangle')^2-4\triangle.f'=0$. Hence the geometrical interpretation of the equation is:—

If from any point six tangents are drawn to a curve of the third order, their six points of contact lie on a conic

* Elliot, Algebra of Quantics, § 77.

$\triangle=0$, *and the six remaining points, in which these tangents meet the cubic again, lie on another conic* $(\triangle')^2-4\triangle.f'=0$, *having double contact with the first at the points where* $\triangle'=0$ *meets* $\triangle=0$.

If, however, (x', y', z') is a point on the cubic, $f'=0$. The discriminant equated to zero gives $\triangle^2=4\triangle'.f$, which is of the fourth degree in (x, y, z). Hence, *only four tangents can be drawn to a cubic from any point on the same.*

It has already been said that the roots of the equation (1) of §63 gives the points where the line meets the curve. Hence, from the conditions that the equation has one or more pairs of double roots, or has two or more roots coincident, it will be possible to obtain bitangents or other multiple tangents of the curve; but the investigation by this method is by no means simple, and we shall see later that these can be obtained by other simpler methods.

68. Geometrical Interpretation:

If $\triangle'f'=0$, it follows from the equation (2) of §63 that the sum of the roots vanishes,

i.e.,
$$\Sigma\frac{\mu}{\lambda}=0, \quad \text{or, } \Sigma\frac{\text{QA}}{\text{PA}}=0,$$

where $\text{A}_1, \text{A}_2, \text{A}_3, \dots \dots \text{A}_n$ are the n points in which the line PQ intersects the curve. If now we put

$$\text{PA}_1=r_1,\ \text{PA}_2=r_2,\ \dots\ \text{PA}_n=r_n, \text{ and } \text{PQ}=\text{R}, \text{ then}$$

$$\text{QA}_1=\text{PQ}-\text{PA}_1=\text{R}-r_1;\ \text{QA}_2=\text{PQ}-\text{PA}_2=\text{R}-r_2, \text{ etc.}$$

Then, $$\Sigma\frac{\mu}{\lambda}=\Sigma\frac{\text{QA}}{\text{PA}}=\Sigma\frac{\text{PQ}-\text{PA}}{\text{PA}}=0, \quad i.e., \quad \Sigma\frac{\text{R}-r}{r}=0,$$

or,
$$\frac{n}{\text{R}}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\dots+\frac{1}{r_n}.$$

The geometrical interpretation of this property has been given by Cotes (1722) in his *Harmonia Mensurarum* as follows:—

If on each radius vector through a fixed point P, there be taken a point Q, such that

$$\frac{n}{PQ}=\frac{1}{PA_1}+\frac{1}{PA_2}+\ldots+\frac{1}{PA_n},$$

where $A_1, A_2, \ldots A_n$ are the points of intersection with the curve, then the locus of Q will be a right line; or, in other words,—

The locus of the Harmonic Mean Q of the points of intersection with a curve of all lines of a pencil, drawn through a fixed point P, is the polar line of P. Similarly for the polar conic. The point Q is called the *Harmonic Mean.*

In general, we call a point Q the *"Harmonic Mean of order k,"* which satisfies the equation

$$\Sigma\frac{QA_1}{PA_1},\ \frac{QA_2}{PA_2}\cdots\cdots\frac{QA_k}{PA_k}=0;$$

thus we can say that the harmonic mean of the kth order lies on the kth polar, and therefore it geometrically signifies the vanishing of the $(k+1)$th term in the equation (2) of §63.

If the pole P is at infinity, then Q is called the *Centre of Mean Distances*, and the several polars are called the *Diametral Curves.*

Ex. 1. If the given curve be a conic, it has only one polar line, and this is the locus of a point Q which is the harmonic conjugate of P *w.r.t.* A_1 and A_2.

Ex. 2. If the point P lies on a cubic curve, then the two points A_1 and A_2, where any line through P meets the cubic, are determined by the two values of λ/μ given by

$$\lambda^2 f + \lambda\mu \triangle f + \mu^2 \triangle' f = 0.$$

If now the line meets the polar conic of P in Q, $\triangle f = 0$, and consequently we have

$$\frac{QA_1}{PA_1} + \frac{QA_2}{PA_2} = 0$$

i.e.,
$$\frac{PQ - PA_1}{PA_1} + \frac{PQ - PA_2}{PA_2} = 0$$

i.e.,
$$\frac{2}{PQ} = \frac{1}{PA_1} + \frac{1}{PA_2}$$

which shows that P, A_1, Q, A_2 form a harmonic range, or in other words,—

The points where any line through a point on a cubic meets the cubic and the polar conic of the point form with the point a harmonic range.

Ex. 3. The polar line of a point at infinity is the diameter of the system of parallel chords directed to that point.

Ex. 4. The polar curves of any point *w.r.t.* a curve are projected into the polar curves of the projected point *w.r.t.* the projection of the given curve.

Ex. 5. A conic touches a cubic at O and cuts it in four other points P, Q, R, S. Show that OP, OQ, OR, OS meet the cubic again in four points lying on a conic, which also touches the cubic at O.

69. The Centre of a Curve :

In a conic the pole of the line at infinity is defined as the *centre*. But in the case of a general curve of order n, a line has $(n-1)^2$ poles, and there is therefore no unique point for such a curve corresponding to the centre of a conic. If, however, the curve be regarded as an envelope, every line has a pole, a polar curve of the second, third and higher class, and finally a polar curve of the $(n-1)$th

class, which is touched by the n tangents at the points where the line meets the curve. Thus, we may obtain a unique point—the pole of the line at infinity—when the curve is defined by its tangential equation.

Let (ξ, η, ζ), (ξ', η', ζ') be the co-ordinates of any two lines. Then the co-ordinates of any line through their intersection, dividing the angle between them in two parts whose sines are in the ratio $\lambda : \mu$, may be taken as

$$(\lambda\xi+\mu\xi', \quad \lambda\eta+\mu\eta', \quad \lambda\zeta+\mu\zeta').$$

If we substitute these values for ξ, η, ζ in the tangential equation of the curve, the equation corresponding to (1) of §63 now determines the ratios of the sines of the parts into which the angle between the two lines is divided by each of the tangents drawn to the curve through their intersection.

If now P is a variable point, and O a fixed point on any given line whose pole is to be determined, and A_1, $A_2, \ldots A_n$ the points of contact of tangents drawn from P, the pole Q of the line has the property

$$\Sigma\left(\frac{\sin \text{Q PA}_1}{\sin \text{A}_1 \text{ PO}}\right)=0 \qquad \ldots \quad (1)$$

For the conic, regarded as an envelope of the second class, this relation becomes

$$\frac{\sin \text{QPA}_1}{\sin \text{A}_1\text{PO}}+\frac{\sin \text{QPA}_2}{\sin \text{A}_2\text{PO}}=0 \qquad \ldots \quad (2)$$

In the language of geometry this may be stated as follows:—

If from any point P on a fixed line OP, tangents PA_1, PA_2 are drawn to a conic, and a line PQ such that

$\{P.OA_1QA_2\}$ is harmonic, then the line OQ passes through a fixed point.*

The relation (1) may be written in the form

$$\Sigma \frac{M_1A_1}{A_1N_1}=0 \qquad \dots \quad (3)$$

where M_1 is the foot of the perpendicular drawn from A_1 on the line PQ, and N_1 is the foot of the perpendicular from the same point on the line OP.

If the line OP now moves off to infinity, then all the denominators in (3) tend to equality, and we have $\Sigma M_1A_1=0$, *i.e.*, the sum of the perpendiculars drawn from the points of contact of any system of parallel tangents on a parallel line through Q is zero.

Hence we have the following definition for the Centre of a Curve, given by Chasles :—†

The centre of mean distances of the points of contact of any system of parallel tangents to a given curve is a fixed point, which may be regarded as a centre of the curve.

Thus, the middle point of the line joining the points of contact of parallel tangents to a curve of the second class (a conic) is a fixed point.

Similarly, in a curve of the third class the centre of gravity of the triangle, formed by the points of contact of three parallel tagents, is a fixed point, and so on.

It is to be noticed, however, that in the system of Cartesian point co-ordinates, when the equation of the curve contains only terms of odd, or only terms of even degree, the curve is symmetrical about the origin, and it may be brought to self-coincidence by rotation through

* Salmon, Conics, §57.

† Quetelet, Correspondence Mathématiques et Physique, VI, 8.

180° about the origin. The origin may, in this case, be called a *centre* of the curve.

Ex. 1. If P is a point on a curve and $PP_1P_2...$a secant cutting the curve in $P_1, P_2,...P_{n-1}$, and the polar conic of P in Q, then

$$\frac{n-1}{PQ}=\Sigma\frac{1}{PP_1}.$$

Ex. 2. If P be a point of inflexion on a curve and $PP_1P_2...P_{n-1}$ be a secant cutting the harmonic polar in Q, prove that

$$\frac{n-1}{PQ}=\Sigma\frac{1}{PP_1}.$$

Ex. 3. If a curve has a centre, the polar curves of the centre have this point for a centre.

Ex. 4. The polar curves of any point at infinity, in Ex. 3, have the centre of the curve as a centre.

Ex. 5. If an n-*ic* $f(x,y)=0$ has a centre, all the partial derivatives of f *w.r.t.* x and y vanish at that point.

[When the origin is taken at the centre (x', y'), the transformed equation becomes $f(x+x', y+y')=0$, and the terms of orders $(n-1)$, $(n-3)$, $(n-5)$... must be absent from this new equation.]

70. MacLaurin's Theorem:

If through any point P a line be drawn meeting the curve in n points, and at these points tangents be drawn and if any other line through P cut the curve in A_1, A_2, $A_3,...A_n$, and the system of n tangents in B_1, $B_2,...B_n$, then

$$\Sigma\frac{1}{PA}=\Sigma\frac{1}{PB}.$$

Consider two lines drawn through P which intersect two curves of order n in the same points R_1, R_2, $R_3...R_n$ and S_1, S_2, $S_3......S_n$ respectively. The polar line of P with respect to both curves must be the same, since

the two harmonic means R and S are the same for both. Now, if PR and PS coincide, the two curves touch each other at n collinear points—R_1, R_2, R_3,.......R_n, but still the polar line of P for both is the same. The tangents at the n points R_1, R_2, R_3,.......R_n may be taken to constitute a curve of the nth order touching the other original curve at n collinear points, and therefore, if a line through P intersect the curve in A_1, A_2, A_3......A_n and the tangents at B_1, B_2, B_3......B_n, the harmonic means of the two systems are the same, and consequently we have

$$\Sigma \frac{1}{PA} = \Sigma \frac{1}{PB}.$$

Ex. 1. The tangent drawn from any point on the polar line of a point P, *w.r.t.* a conic is cut harmonically by the tangents drawn from P.

Ex 2. A radius vector drawn through a point O on a cubic meets the curve again in P and Q. Shew that the locus of the extremities of harmonic means between OP and OQ is a conic, which reduces to a right line when Q is a point of inflexion.

71. Polar Curves of the Origin:

Let the Cartesian equation of a curve be

$$u_0 + u_1 + u_2 + \ldots\ldots + u_n = 0$$

or, this may be written in the homogeneous form:

$$f \equiv u_0 z^n + u_1 z^{n-1} + u_2 z^{n-2} + \ldots\ldots + u_n = 0.$$

The co-ordinates of the origin may be taken as (0, 0, 1).

$$\therefore \quad \triangle \equiv \left(0.\ \frac{\partial}{\partial x} + 0.\ \frac{\partial}{\partial y} + 1.\ \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}$$

Therefore, the different polar curves are respectively—

$$\frac{\partial f}{\partial z}=0, \quad \frac{\partial^2 f}{dz^2}=0, \ldots\ldots \quad \frac{\partial^{n-1} f}{\partial z^{n-1}}=0.$$

The polar line of the origin becomes—

$$\frac{\partial^{n-1} f}{\partial z^{n-1}}=n\,!\; u_0\, z+(n-1)\,!\; u_1=0, \text{ i.e., } nu_0+u_1=0.$$

The polar conic is—

$$\frac{\partial^{n-2} f}{\partial z^{n-2}}=\frac{n\,!}{2\,!}u_0z^2+(n-1)\,!\; u_1z+(n-2)\,!\; u_2$$

$$=n(n-1)u_0+2(n-1)u_1+2u_2=0.$$

The polar cubic is—

$$\frac{\partial^{n-3}}{\partial z^{n-3}} f=\frac{n\,!}{3\,!}u_0z^3+\frac{(n-1)\,!}{2\,!}u_1z^2$$

$$+(n-2)\,!\; u_2\, z+(n-3)\,!\; u_3$$

$$=n(n-1)(n-2)u_0+3(n-1)(n-2)u_1$$

$$+6(n-2)u_2+6u_3=0$$

etc., etc., etc.

From these the polar curves of any point on a curve can easily be found; the point may be taken as the origin of a system of Cartesian axes and the corresponding equation of the given curve may be obtained by the usual method.

72. From a study of these equations we at once draw the following inferences :—

(1) If the origin lies on the curve, $u_0=0$, all the polar curves pass through the origin, and $u_1=0$ is the common tangent to all.

Thus, the polar curves of any point on the curve pass through the point and have a common tangent there, *i.e.*, the polar curves all touch the curve at that point.

(2) If the origin is a double point on the curve, the first degree terms are absent from the equation, and it is found that they are absent also from the equations of all polars. The terms of the lowest degree are u_2 in all of them.

Hence we infer that all the polar curves of a double point on the curve have a double point with the same tangents at the double point on the original curve.

Further, the polar conic of the double point is $u_2=0$, which represents the two tangents at that point. Hence the polar conic of a double point on the curve breaks up into two right lines.

(3) In general, if the lowest terms in the equation of a curve are u_k, the lowest terms in all the polar curves are u_k.

Hence we infer that all the polar curves of any multiple point of order k on the curve have a multiple point of the same order with the same tangents at that point.

(4) If the origin is a point of inflexion on the curve, the linear terms are a factor of the second degree terms, and hence, from the equations of §71, it follows that the origin is an inflexion on all the polar curves; and similarly in general.

Thus, the polar curves of a point on the curve at which the tangent has r-pointic contact have r-pointic contact at the point with the same tangent.

Ex. 1. The $(n-k)$th polar of a k-ple point of an n-*ic* consists of the k tangents at the point.

Ex. 2. If P lies on a given m-*ic*, the envelope of the polar line of P with respect to a given n-*ic* is of class $m(n-1)$.

Ex. 3. The kth polar of P with respect to an n-*ic* having an $(n-1)$-ple point at O is an $(n-k)$-*ic* having an $(n-k-1)$-ple point at O.

73. *If a point* Q (x'', y'', z'') *lies on the k-th polar of a point* P (x', y', z'), *then* P *lies on the* $(n-k)$*th polar of* Q *with respect to a given curve.*

This is only a geometrical statement of the equations in §63. For, the kth polar of P is—

$$\left(x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right)^k f=0$$

or,
$$\left(x \frac{\partial}{\partial x'} + y \frac{\partial}{\partial y'} + z \frac{\partial}{\partial z'} \right)^{n-k} f'=0.$$

If Q lies on this, we must have

$$\left(x'' \frac{\partial}{\partial x'} + y'' \frac{\partial}{\partial y'} + z'' \frac{\partial}{\partial z'} \right)^{n-k} f'=0.$$

Therefore, the locus of P (x', y', z') is—

$$\left(x'' \frac{\partial}{\partial x} + y'' \frac{\partial}{\partial y} + z'' \frac{\partial}{\partial z} \right)^{n-k} f=0$$

which is the $(n-k)$th polar of Q.

74. *The locus of all points* whose *polar lines pass through a fixed point is the first polar of that point.*

The polar line of any point (x', y', z') is

$$\left(x \frac{\partial}{\partial x'} + y \frac{\partial}{\partial y'} + z \frac{\partial}{\partial z'} \right) f'=0.$$

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If this passes through a fixed point (x'', y'', z''), we must have

$$x''\frac{\partial f'}{\partial x'} + y''\frac{\partial f'}{\partial y'} + z''\frac{\partial f'}{\partial z'} = 0$$

which shows that the locus of (x', y', z') is the curve $\triangle f = 0$, which is the first polar of (x'', y'', z'').

In a like manner, it can be shown that the locus of points whose polar conics pass through a given point is the second polar of the point; and in general, the locus of points whose kth polars pass through a fixed point is the $(n-k)$th polar of the point.

75. *Every point has only one polar line.*

For, the polar line of the point (x', y', z') is—

$$x\frac{\partial f'}{\partial x'} + y\frac{\partial f'}{\partial y'} + z\frac{\partial f'}{\partial z'} = 0.$$

The co-efficients in this equation are known, determinate functions of (x', y', z'), and therefore the line is determinate and unique. Consequently, there is only *one* polar line of a given pole.

76. *The first polar of every point on a right line passes through the pole of that line.*

The polar line of any point (x', y', z') is—

$$x\frac{\partial f'}{\partial x'} + y\frac{\partial f'}{\partial y'} + z\frac{\partial f'}{\partial z'} = 0.$$

If (x'', y'', z'') is a point on this line, we must have—

$$x''\frac{\partial f'}{\partial x'} + y''\frac{\partial f'}{\partial y'} + z''\frac{\partial f'}{\partial z'} = 0$$

which shows that the first polar of (x'', y'', z'') passes through the point (x', y', z'), which is the pole.

77. *There are* $2(n-2)$ *points on a right line, the first polars of which touch the line. The polar conics of their* $2(n-2)$ *points of contact also touch the same line.*

Let (x', y', z') be any point on the line $lx+my+nz=0$ (1)

The first polar of (x', y', z') *w.r.t.* $f=0$ is—

$$F \equiv x' \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + z' \frac{\partial f}{\partial z} = 0 \qquad \ldots \quad (2)$$

The tangent at a point (x'', y'', z'') on $F=0$ is—

$$x \frac{\partial F}{\partial x''} + y \frac{\partial F}{\partial y''} + z \frac{\partial F}{\partial z''} = 0,$$

$$i.e., \; x(x'f''_{11} + y'f''_{12} + z'f''_{13}) + y(x'f''_{12} + y'f''_{22} + z'f''_{23})$$
$$+ z(x'f''_{13} + y'f''_{23} + z'f''_{33}) = 0 \qquad \ldots \quad (3)$$

where $f'' \equiv f(x'', y'', z'')$, and f_{11}, f_{12}, f_{13}, etc., denote the second differential co-efficients of f.

If this is to be the same as the line (1), we must have

$$\left.\begin{aligned} x'f''_{11} + y'f''_{12} + z'f''_{13} &= kl \\ x'f''_{12} + y'f''_{22} + z'f''_{23} &= km \\ x'f''_{13} + y'f''_{23} + z'f''_{33} &= kn \\ \text{also} \qquad lx'' + my'' + nz'' &= 0 \end{aligned}\right\} \qquad \ldots \quad (4)$$

Eliminating k and (x'', y'', z'') between the equations (4), we shall obtain the locus of (x', y', z'). Now the eliminant* will be of degree $2(n-2)$ in the variables x', y', z', which therefore represents a $2(n-2)$-*ic*, and the points

* Clebsch, Leçons sur la Geometrie, Tom II, p. 13.

where $lx+my+nz=0$ intersects the locus are the required points. Hence the truth of the theorem follows.

If, however, we eliminate k and (x', y', z') between the first three equations in (4) and $lx'+my'+nz'=0$, we obtain the locus of (x'', y'', z'') in the form of a determinant equation, namely,

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} & l \\ f_{12} & f_{22} & f_{23} & m \\ f_{13} & f_{23} & f_{33} & n \\ l & m & n & 0 \end{vmatrix}=0 \qquad \ldots \quad (5)$$

Equation (5) is of degree $2(n-2)$ in the variables, and therefore the given line intersects this curve in $2(n-2)$ points which are the points of contact of the first polars with the given line.

Again, the polar conic of (x'', y'', z'') is—

$$f''_{11}x^2+f''_{22}y^2+f''_{33}z^2+2f''_{23}yz+2f''_{31}zx+2f''_{12}xy=0.$$

If this touches the line $lx+my+nz=0$, we must have

$$\begin{vmatrix} f''_{11} & f''_{12} & f''_{13} & l \\ f''_{21} & f''_{22} & f''_{23} & m \\ f''_{31} & f''_{32} & f''_{33} & n \\ l & m & n & 0 \end{vmatrix}=0$$

which is satisfied, since (x'', y'', z'') is a point on the curve (5).

From what has been said above, we may deduce the following theorem:

Through any point we may draw two lines, on each of which there is a point whose first polar touches the line at the given point. These are the two tangents which can be drawn from the given point to its polar conic.

These two tangents may coincide either when the point lies on the polar conic, in which case the point must lie on the original curve, or when the polar conic breaks up into two right lines, and the tangents coincide with the line joining the point to the intersection of those two lines. In this case, as we shall show later on, the point lies on the Hessian.

From the theorems just stated it will be seen that there is somewhat of a reciprocal relation between the first polar and the polar conic of a point, as will appear in the sequel.

78. *Every straight line has, in general* $(n-1)^2$ *poles.*

For, take any two points A and B on the line. The first polar of each of these points passes through the pole of the line (§ 76). Therefore the points of intersection of these two first polars are the poles of the line. But each of these curves is of the $(n-1)th$ degree, and they intersect, in general, in $(n-1)^2$ points, which are the poles of the line.

Cor. The first polars of all points on the line pass through these $(n-1)^2$ poles. For, if P and Q be the first polars of A and B, then the first polar of any point on AB is $P+\lambda Q=0$, which evidently passes, for all values of λ, through all the intersections of P and Q.

79. If, however, the curve has a node, the first polar of every point passes through it (§ 65), and therefore the two first polars intersect in only $(n-1)^2-1$ other points, which are the poles of the line. Therefore, if the curve has δ nodes, the first polars intersect in only $(n-1)^2-\delta$ other points, which are the poles of the line.

If the curve has a cusp, the first polar of any point not only passes through it, but also touches the cuspidal tangent.* Therefore the cusp counts as two among the

* To be proved hereafter in § 85.

intersections of the two first polars, which then intersect only in $(n-1)^2-2$ other points, and these are the poles of the line. If the curve has κ cusps, the two first polars intersect only in $(n-1)^2-2\kappa$ other points, which are the poles of the line.

Hence, combining all these results, we may enunciate the following theorem :—

Every right line has, in general $(n-1)^2$ *poles with regard to a non-singular curve, but if the curve has* δ *nodes and* κ *cusps, the number of poles is reduced to* $(n-1)^2-\delta-2\kappa$.

Ex. The pole of the line $lx+my+nz=0$ with regard to the degenerate cubic $xyz=0$ is the point $(1/l, \ 1/m, \ 1/n)$.

[The point and the line are called the *pole* and *polar w.r.t.* the triangle.]

80. *If the polar line (or any other polar curve) of a point passes through the point, the point lies on the curve.*

For, if we put (x, y, z) for (x', y', z') in the equation of the polar, it becomes identical with the equation of the curve, since the polar curves are obtained by operation with $\triangle$, which becomes in this case

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},$$

the effect of which, by Euler's Theorem, on a homogeneous function is only to multiply the function by a numerical factor.

It is to be noted that the polar line of every point on a curve is the tangent at that point.

81. The converse theorem is also true (§72), *i.e., every polar curve of a point on the curve touches the curve at that point.*

Let F_r be the rth polar of any point (x', y', z'), so that its equation is given by—

$$F_r = \left(x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right)^r f = 0 \qquad \dots \quad (1)$$

which evidently passes through the point (x', y', z'), since the point lies on the curve.

Now, the tangent at (x', y', z') to the rth polar F_r is—

$$\left(x \frac{\partial}{\partial x'} + y \frac{\partial}{\partial y'} + z \frac{\partial}{\partial z'} \right) F_r' = 0 \qquad \dots \quad (2)$$

But F_r is of degree $n-r$, and therefore, by the identities of §63, equation (2) may be written as—

$$\left(x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right)^{n-r-1} F_r = 0$$

i.e., as $\qquad \left(x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right)^{n-1} f = 0$

or, $\qquad \left(x \frac{\partial}{\partial x'} + y \frac{\partial}{\partial y'} + z \frac{\partial}{\partial z'} \right) f' = 0$

which is the equation of the tangent at (x', y', z') to the curve $f=0$.

82. *The points of contact of tangents drawn to a curve from any point lie on the first polar of that point.*

Let (x', y', z') be a point on the curve. The tangent at (x', y', z') is—

$$x \frac{\partial f'}{\partial x'} + y \frac{\partial f'}{\partial y'} + z \frac{\partial f'}{\partial z'} = 0.$$

If this passes through any point (x'', y'', z''), we must have—

$$x'' \frac{\partial f'}{\partial x'} + y'' \frac{\partial f'}{\partial y'} + z'' \frac{\partial f'}{\partial z'} = 0,$$

which shows that the locus of (x', y', z') is—

$$x'' \frac{\partial f}{\partial x} + y'' \frac{\partial f}{\partial y} + z'' \frac{\partial f}{\partial z} = 0;$$

and this is the first polar of the point (x'', y'', z'').

From this it follows that the points of contact of the tangents drawn from a given point to the curve are the points of intersection of the curve with the first polar of the point. Now, the first polar of any point is of degree $(n-1)$, and consequently it intersects the curve of the nth degree in $n(n-1)$ points. Thus we see that *from a given point there can be drawn, in general, $n(n-1)$ tangents to a curve of the nth degree.*

Definition: The *Class* of a curve is determined by the number of tangents which can be drawn from a given point to the curve, and will usually be denoted by m.

The above theorem can therefore be stated as follows:—

The class of a curve is, in general, $n(n-1)$, or as we shall see later on, *the degree of the reciprocal polar curve is $n(n-1)$.*

83. Let us examine the case when the given point lies on the curve. We have seen (§72) that the first polar touches the curve at the given point. Hence that point counts as two of the intersections of the first polar with the curve. Therefore, the number of tangents (different from the tangent at the point) which can be drawn from the point to touch the curve elsewhere is $n(n-1)-2$, or, $(n+1)(n-2)$. Hence we obtain the theorem:—

From any point on a curve not more than $(n+1)(n-2)$, or, $m-2$ tangents (excluding the tangent at the point) can be drawn to the curve.

If the tangent at a point has a contact of the second order with the curve, *i.e.*, if it be a point of inflexion on the curve, the tangent is to be regarded once as the tangent at the point and once as one of the tangents which can be drawn from the point to the curve. Therefore, there can be drawn only $(n+1)(n-2)-1$, or, $n(n-1)-3$ other tangents to the curve.

Hence we obtain the theorem :—

From a point of inflexion on a curve, only $n(n-1)-3$, *or,* $m-3$ *tangents can be drawn to touch the curve elsewhere.*

84. If, however, the point is a double point on the curve, we shall have to distinguish between the cases when (1) it is a *node*, (2) it is a *cusp*.

Suppose the point is a node on the curve. We have seen that the first polar of a node has a node with the same nodal tangents at the point. Therefore, the double point counts as six among the intersections of the first polar with the curve (§51). Of these, however, two belong to the two nodal tangents.

Thus the *number of tangents, exclusive of the nodal tangents, which can be drawn from a node to touch a curve elsewhere is* $n(n-1)-4$, *or,* $m-4$.

Next, suppose that the point is a cusp on the curve. The first polar has the cuspidal tangent as a tangent. Hence the point counts as three among the intersections, besides the two which belong to the cuspidal tangent.

Therefore *the number of tangents, exclusive of the cuspidal tangent, which can be drawn from a cusp to touch a curve elsewhere is* $n(n-1)-3$, *or,* $m-3$.

In general, if the point is a multiple point of order k, there are k tangents at the point, each of which counts as two among the tangents which can be drawn from the point to the curve. Thus the k tangents count as $2k$ tangents drawn from the point, and $n(n-1)-2k$ or $m-2k$ other tangents can be drawn, distinct from the tangents at the multiple point, to touch the curve elsewhere.

Ex. 1. Prove that the points of contact of tangents drawn from the point (h, k) to the curve $x^3+y^3=3axy$ lie on a conic through the origin.

Discuss the case when the point is at the origin.

Ex. 2. Show that the conic in Ex. 1 will bisect the angle between the nodal tangents, if $h+k=0$.

Ex. 3. If the tangent at any point (x_1, y_1) on the cubic $x^3+y^3=a^3$ meets it again in (x_2, y_2), show that $x_2/x_1+y_2/y_1+1=0$.

85. *The first polar of a point passes through every double point and its tangent at that point is harmonic conjugate of the line joining the double point with the pole with respect to the nodal tangents of the curve.*

The equation of a curve having the origin for a node, with the axes of x and y as tangents, may be written as

$$f(x, y) \equiv xy + u_3 + u_4 + \dots\dots + u_n = 0$$

or, in the homogeneous form—

$$f(x, y, z) \equiv xyz^{n-2} + u_3 z^{n-3} + u_4 z^{n-4} + \dots\dots + u_n = 0.$$

The first polar of any point (x', y', z') is—

$$\mathrm{F}(x, y, z) \equiv (x'y + y'x)z^{n-2} + \text{lower powers of } z = 0,$$

which shows that the first polar passes through the origin (double point), and that its tangent at that point is—

$$x'y + y'x = 0 \qquad \dots \quad (1)$$

Now, the line joining the point (x', y', z') to the origin is

$$x'y - y'x = 0 \qquad \ldots \quad (2)$$

The lines (1) and (2) are evidently harmonic conjugates with respect to the two axes, which are the nodal tangents.*

If, however, the origin is a cusp, with the axis of y as the cuspidal tangent, the first polar becomes—

$$F(x, y, z) \equiv (x' + y')xz^{n-2} + \text{lower powers of } z = 0.$$

$\therefore$ The axis of $y(x=0)$ is also a tangent to the first polar, *i.e.*, the first polar of any point touches the cuspidal tangent and meets the curve *three times* at a cusp.

From what has been said in §82, it follows that the first polar of a point passes through all the double points, etc., and the points of contact of tangents drawn from the given point to the curve.

Ex. 1. Show that the points of contact of tangents drawn from the point (a, b) to the curve $bx^3 + ay^3 = 1$ lie on a circle.

Ex. 2. The locus of points of contact of tangents drawn from any point to the system of curves $y = \lambda x^3$ is a rectangular hyperbola.

Ex. 3. Show that all the polar conics of the curve $y = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$ are parabolas.

Ex. 4. Show that the points of contact of tangents drawn from the origin to the curve $xy = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$ are collinear.

86. *If the curve has a multiple point of order k, that point will be a multiple point of order $k-1$ on the first polar, of order $k-2$ on the second polar, and so on.*

Let the origin be a multiple point of order k on the curve. The equation of the curve is therefore of the form—

$$f \equiv u_k + u_{k+1} + u_{k+2} \ldots\ldots + u_n = 0$$

* Salmon, Conic Sections, §57.

where u_r is a binary r-*ic* in x and y, or, in the homogeneous form—

$$f \equiv u_k z^{n-k} + u_{k+1} z^{n-k-1} + \ldots\ldots\ldots + u_n = 0.$$

The first polar of any point (x', y', z') is—

$$\mathrm{F} \equiv x' \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + z' \frac{\partial f}{\partial z} = 0 \qquad \ldots \quad (1)$$

The lowest terms in $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are of degree $(k-1)$ and those in $\dfrac{\partial f}{\partial z}$ are of degree k in x and y. Therefore, the lowest terms in x and y being of degree $k-1$ in (1), the origin is a multiple point of order $k-1$ on the curve F.

Similarly, the lowest terms in the equation of the second polar are of degree $k-2$ in x and y, and so on, which proves the proposition.

87. It appears from the above that if u_k has a square factor u_1, *i.e.*, if $u_k = u_1^2 u_{k-2}$, then this factor occurs also in the lowest terms in the equation of the first polar.

For, $$\frac{\partial u_k}{\partial x} = \frac{\partial}{\partial x}(u_1^2 u_{k-2})$$

$$= u_1^2 \frac{\partial u_{k-2}}{\partial x} + 2u_1 u_{k-2} \frac{\partial u_1}{\partial x}$$

$$= u_1 \left\{ u_1 \frac{\partial u_{k-2}}{\partial x} + 2u_{k-2} \frac{\partial u_1}{\partial x} \right\}$$

Similarly, $$\frac{\partial u_k}{\partial y} = u_1 \left\{ u_1 \frac{\partial u_{k-2}}{\partial y} + 2u_{k-2} \frac{\partial u_1}{\partial y} \right\}$$

Also the lowest terms in $\dfrac{\partial f}{\partial z}$ contains u_1^2 as a factor.

Thus the lowest terms in the equation of the first polar contain u_1 as a factor, and therefore $u_1=0$ is a tangent to the first polar. Hence we obtain the following theorem :—

If two tangents at a multiple point coincide, the coincident tangent touches the first polar of every point.

In particular, the first polar of any point touches the cuspidal tangent, as proved in §85.

Again, if u_k has any factor u_1 in the lth degree, *i.e.*, if $u_k \equiv u_1{}^l . v_{k-l}$, u_1 will occur in the $(l-1)$th degree in the lowest terms of the equation of the first polar, in degree $(l-2)$ in those of the second polar, and so on.

Hence we conclude that, in general, if l tangents at the k-ple point on a curve coincide, the coincident tangent will appear as $(l-1)$ coincident tangents at the multiple point of the first polar, as $(l-2)$ coincident tangents at the multiple point of the second polar, and so on.

Ex. 1. Show that the points of contact of tangents drawn from the point $(0, -1, 1)$ to the cubic $x^3+y^3+z^3+6mxyz=0$ are collinear on the line $y-z=0$.

Ex. 2. The points of contact of parallel tangents to a curve of the nth degree lie on a curve of the $(n-1)$th degree. (Serret.)

(They lie on the first polar of a point at infinity.)

Ex. 3. The polar conics of two points A, B with regard to a cubic curve are rectangular hyperbolas. Prove that the line AB has four poles, any one of which is the orthocentre of the triangle formed by the other three.

Ex. 4. Prove that the properties of polar curves are unaltered by projection.

Ex. 5. Show that the envelope of the polar lines of points on a given line *w. r. t.* an n-*ic* is of class $(n-1)$.

Ex. 6. Show that the tangents to the curve $x^m y^n = a^{m+n}$ drawn from the point (α, β) touch the same at points lying on the hyperbola

$$(m+n)xy = n\beta x + m\alpha y.$$

CHAPTER V

COVARIANT CURVES—THE HESSIAN, THE STEINERIAN AND THE CAYLEYAN

88. In this Chapter we shall discuss the properties of three covariant curves—the *Hessian*, the *Steinerian* and the *Cayleyan*—which are geometrically associated with a given curve and can be derived from it by geometrical processes. Professor J. Steiner * in a paper in 1854 discussed a number of general properties of these curves which were, however, studied in detail by subsequent writers. Prior to him, Hesse† studied the properties of the first curve, which is named after him—the Hessian, and the third one was studied by Cayley for a curve of the third order.‡ Cremona calls the second and the third curves the Steinerian and the Cayleyan respectively of the given curve.

In the second fundamental theorem, as he calls it, Steiner states a number of properties of polar curves ; the following, among others, is of special importance :

If the kth polar of a point A w.r.t a given curve has a double point at another point B, then the $(n-k-1)$th polar of B has a double point at A.

From this again, he enunciates the two theorems :

The locus of a point whose first polar has a double point is a curve of order $3(n-2)^2$, and the locus of

* J. Steiner—*Allgemeine Eigenschaften der algebraischen Curven*—Crelle, Bd. 47 (1854), pp. 1-6. (Abgedruckt aus dem Monatsbericht der hiesigen Akademie der Wissenschaften vom August, 1848.)

† Dr. Otto Hesse—Über die Elimination, etc.,—Crelle, Bd 28 (1844), pp. 68-107.

‡ Cayley—Memoire sur les courbes du troisième ordre —Journal de Mathematiques pures et appliques (Paris), Vol. (1) 9 (1844), pp. 285-93, or, Coll. Works—Vol. I, p. 183 and Vol. II, p. 385.

the double point is a curve of order 3 $(n-2)$, which is consequently the locus of points whose $(n-2)$th polar has a double point. These two loci are called by him "Conjugirte Kern-Curven." The line joining the two points A and B envelopes a third curve, which is of the same class as the locus of A.

The locus of A is called the *Steinerian*, that of B, the *Hessian* while the envelope of AB is called the *Cayleyan*. The nodal tangents of the first polars, again, envelope a fourth curve. The points A and B are said to be *corresponding* points. We shall now proceed to consider the properties of the first three curves,* one after the other.

Note : The Hessian passes through the points of inflexion, while the Steinerian and the Cayleyan touch the inflexional tangents of the given curve.

89. Let f be any quantic in the variables x, y, z, and let f_1, f_2, f_3 denote its first differential co-efficients with regard to x, y, z respectively. If f_{11}, f_{12}, f_{13}, etc., denote the second differential co-efficients of f with respect to the same variables, then the determinant—

$$H = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

is called the *Hessian*† of f.

* These curves were studied geometrically by Cremona in his *Introduzione ad una teoria geom delle curve piane*, and analytically by Clebsch—Crelle, Bd. 59 (1861), p. 125, and Bd. 64 (1865), pp. 288-93.

† It is called the Hessian, because it was first studied by the German Mathematician Dr. Otto Hesse. It is a *covariant* function of f (Elliot, Algebra of Quantics, § 11), and has important applications in the theory of curves. The locus denoted by $H=0$ is called the Hessian of the curve $f=0$.

90. The Hessian :

The Hessian of a curve is the locus of points whose polar conics break up into two right lines.

The polar conic of any point (x', y', z') is $\Delta'^2 f=0$, or,

$$a' x^2+b' y^2+c' z^2+2f' yz+2g'zx+2 h'xy=0$$

where a', b', c', etc., denote the second differential co-efficients $f'_{11}, f'_{22}, f'_{33}$, etc., with respect to x', y', z'.

If this breaks up into two right lines, its discriminant must vanish, *i.e.*, we must have—

$$\begin{vmatrix} f'_{11} & f'_{12} & f'_{13} \\ f'_{21} & f'_{22} & f'_{23} \\ f'_{31} & f'_{32} & f'_{33} \end{vmatrix}=0$$

Therefore the locus of the point (x', y', z') is—

$$H\equiv\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}=0$$

which is called the Hessian of the curve $f=0$.

Since each of the functions f_{11}, f_{12}, etc., is of order $(n-2)$ in the variables, and the determinant is of order three in the functions f_{11}, f_{12}, etc., the determinant equation is of the $3(n-2)$th order in the variables. Thus the degree of the Hessian is $3(n-2)$.

Ex. 1. Find the Hessians of the following curves :

(*i*) $x^3+y^3=xy(x+y+z)$ (*ii*) $(x+y+z)^3+6kxyz=0$

((*iii*) $(yz+x^2)^2=xy^3$.

Ex. 2. Find an expression for the Hessian of a curve of order n defined by the explicit equation—

$$y=f(x).$$

Ex. 3. Show that the Hessian of a cubic is also a cubic curve.

Ex. 4 Find the Hessian of the curve defined by its polar equation

$$f(r, \phi)=0.$$

Ex. 5. Show that the class of the Hessian of a curve having no singular point is $3(n-2)(3n-7)$.

91. Proceeding as in the preceding Article, we may obtain the locus of points whose polar conics are—

(*i*) *Parabolas.* (*ii*) *Rectangular Hyperbolas.*

(*i*) The polar conic of the point (x', y', z') being

$$a' x^2+b' y^2+c' z^2+2f'yz+2g'zx+2h'xy=0 \qquad \ldots \quad (1)$$

the condition* that this represents a parabola is—

$$\begin{vmatrix} a' & h' & g' & a \\ h' & b' & f' & b \\ g' & f' & c' & c \\ a & b & c & 0 \end{vmatrix}=0$$

where $a', b', c', \ldots$ have the significance of § 90, and a, b, c denote the sides of the fundamental triangle.

Thus the locus of points whose polar conics are parabolas is obtained in the form:

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} & a \\ f_{21} & f_{22} & f_{23} & b \\ f_{31} & f_{32} & f_{33} & c \\ a & b & c & 0 \end{vmatrix}=0 \qquad \ldots \quad (2)$$

In Cartesian co-ordinates, however, this equation becomes

$$f_{11}\, f_{22}=(f_{12})^2$$

This curve, denoted by $G=f_{11}.f_{22}-(f_{12})^2=0$, divides the plane into regions ; the polar conics of points in the region in which the expression G is positive, meet the line

* Salmon, Conics, § 285.

at infinity $z=0$ in imaginary points (Ellipses or imaginary lines) and the polar conics of points in the region defined by G negative meet z in real points (hyperbolas and real str. lines). The curve $G=0$ is the *diacritic* or diacritic curve of z with respect to $f=0$. If $f=0$ be a cubic, G is the Poloconic of z. The diacritic is also defined as the locus of points of contact of curves $f_1=$constant with curves $f_2=$const. (Scott—Note on the real inflexions, etc.—Trans. of the Am. Math. Soc., Vol. 3 (1902), pp. 399-400).

The degree of the equation (2) is evidently $2(n-2)$, and therefore the locus is a $2(n-2)$-*ic*. This intersects the n-*ic* in $2n(n-2)$ points, whose polar conics are parabolas. Hence we have the theorem that *on an n-ic, there are $2n(n-2)$ points whose polar conics are parabolas.*

If, however, the n-*ic* has a cusp, its polar conic is the cuspidal tangent taken twice, which is then to be regarded as a degenerate parabola.

Since the locus intersects the n-*ic* in two points at a cusp and in $2n(n-2)-2$ other points, the number of points on an n-*ic* with κ cusps which have non-degenerate parabolas as polar conics becomes $2n(n-2)-2\kappa$.

When the curve is unicursal, and its double points are all cusps, this number becomes

$$2n(n-2)-(n-1)(n-2)=(n-2)(n+1)$$

Proceeding in the same way, it can be shown that there are $(n-2)^2$ points in the plane of an n-*ic* whose polar conics are circles.

In Cartesian co-ordinates the equations giving the points are

$$f_{11}=f_{22} \quad \text{and} \quad f_{12}=0.$$

Ex. 1. The locus of a point whose polar conic with respect to a given n-*ic* touches a given line is a $2(n-2)$-*ic*.

In the case of a cubic, the locus is a conic, which is called the *pole conic* of the line.

Ex. 2. Show that the polar conic of a point with respect to the cubic $x^3+y^3=a(x^2+y^2)$ will be a circle, if the point lies on the line $x=y$.

Ex. 3. If the polar conic of A *w. r. t.* a cubic has its centre at B, the polar conic of B will have its centre at A.

(*ii*) The condition * that the polar conic (1) is a rectangular hyperbola is—

$$\frac{b^2b'+c^2c'-2bcf'}{a^2}\cos A+\frac{c^2c'+a^2a'-2cag'}{b^2}\cos B$$
$$+\frac{a^2a'+b^2b'-2abh'}{c^2}\cos C=0$$

where A, B, C are the angles of the fundamental triangle.

This is a linear function of a', b', c', etc.; and each of these functions being of order $(n-2)$ in the variables, the locus is a curve of order $(n-2)$, which intersects the *n-ic* in $n(n-2)$ points. Hence there are $n(n-2)$ points on an *n-ic* whose polar conics are rectangular hyperbolas. In Cartesian co-ordinates, however, the equation of the locus becomes—

$$f_{11}+f_{22}=0.$$

Ex. The locus of points whose polar conics with respect to a cubic are rectangular hyperbolas is a straight line, but the locus is a conic for points whose polar conics are parabolas.

92. Theorem :

If the first polar of a point A (x', y', z') *has a double point at* B (x'', y'', z''), *then the polar conic of* B *has a double point at* A.

The first polar of A (x', y', z') is—

$$F\equiv x'\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+z'\frac{\partial f}{\partial z}=0$$

If this has a double point at B (x'', y'', z''), the first differential co-efficients of F should vanish at (x'', y'', z'') (§47).

* Askwith—Analytical Geometry of the Conic Sections, § 276.

Therefore, we must have—

$$\frac{\partial F}{\partial x''}=0, \quad \frac{\partial F}{\partial y''}=0, \quad \frac{\partial F}{\partial z''}=0$$

i.e.,

$$\left.\begin{aligned} x'f''_{11}+y'f''_{12}+z'f''_{13}=0 \\ x'f''_{21}+y'f''_{22}+z'f''_{23}=0 \\ x'f''_{31}+y'f''_{32}+z'f''_{33}=0 \end{aligned}\right\} \quad \cdots \quad (1)$$

Again, the polar conic of B (x'',y'',z'') is—

$$C\equiv f''_{11}x^2+f''_{22}y^2+f''_{33}z^2+2f''_{23}yz+2f''_{31}zx+2f''_{12}xy=0.$$

If this has a double point at A (x',y',z'), we must have

$$\frac{\partial C}{\partial x'}=0, \quad \frac{\partial C}{\partial y'}=0, \quad \frac{\partial C}{\partial z'}=0$$

i.e.,

$$\left.\begin{aligned} x'f''_{11}+y'f''_{12}+z'f''_{13}=0 \\ x'f''_{21}+y'f''_{22}+z'f''_{23}=0 \\ x'f''_{31}+y'f''_{32}+z'f''_{33}=0 \end{aligned}\right\} \quad \cdots \quad (2)$$

The conditions (1) and (2) are the same, whence the truth of the theorem follows.

If we eliminate (x',y',z') between the equations (1) or (2), we obtain the locus of B (x'',y'',z''), which is the Hessian of $f=0$.

93. The Steinerian:

If, however, we eliminate (x'',y'',z'') between the same equations, we obtain the locus of (x',y',z'), which is called the *Steinerian*, after the name of the German Mathematician *Steiner*. Thus, the Steinerian is the locus of points whose first polars have double points, or it is the locus of points of intersection of each pair of lines which constitute the polar conic of a point on the Hessian. The degree of the Steinerian is $3(n-2)^2$. For, the resultant of three equations of orders m, n, p in three variables is

of degree np in the co-efficients of the first, of degree mp in those of the second and of degree mn in the co-efficients of the third.* Each of the above three equations is of degree $(n-2)$ in the variables (x'', y'', z''). Therefore the co-efficients of each equation occur in degree $(n-2)^2$ in the resultant. But the co-efficients of each are linear functions of x', y', z'. Thus the resultant is of degree $3(n-2)^2$ in (x', y', z'), and therefore the order of the Steinerian is $3(n-2)^2$. Its class is $3(n-1)(n-2)$.

In the case of a cubic curve, however, the first polar is its polar conic, and therefore the Steinerian $S=0$ and the Hessian $H=0$ coincide, both being the locus of double points of polar conics.

From what has been said above we may enunciate the following definitions :—

The Steinerian $S=0$ is the locus of the double points of polar conics, or, the locus of points whose first polars have double points.

Similarly, *the Hessian $H=0$ is the locus of the double points of first polars, or, the locus of points whose $(n-2)$th polars have double points.*

94. The Steinerian as an Envelope :

The Steinerian † may again be defined as *the envelope of lines two of whose poles coincide, and the locus of these coincident poles is the Hessian.*

Let $\xi x + \eta y + \zeta z = 0$ (1) be a line, two of whose poles coincide at P (x', y', z').

Since the line (1) is the polar line of P, it must be identical with—

$$x \frac{\partial f}{\partial x'} + y \frac{\partial f}{\partial y'} + z \frac{\partial f}{\partial z'} = 0 \qquad \dots \quad (2)$$

* Clebsch—Leçons sur la Géomérie, Tom II, p. 13.

† See Clebsch—" Ueber einige von Steiner behandelte Curven"—Crelle—Bd. 64, pp. 288-90.

Identifying the equations (1) and (2), we obtain

$$\frac{\partial f}{\partial x'} : \frac{\partial f}{\partial y'} : \frac{\partial f}{\partial z'} = \xi : \eta : \zeta$$

Hence, the given line (1) is the polar line of each of the $(n-1)^2$ intersections of the two curves—

$$\xi \frac{\partial f}{\partial z} = \zeta \frac{\partial f}{\partial x} \qquad \dots \quad (3)$$

and

$$\eta \frac{\partial f}{\partial z} = \zeta \frac{\partial f}{\partial y} \qquad \dots \quad (4)$$

which are evidently the first polars of $(\zeta, 0, -\xi)$ and $(0, \zeta, -\eta)$, which are points on the line (1). If two poles of (1) coincide at P, the curve (3) and (4) must touch at P.

Now, the tangents to (3) and (4) at P are—

$$x(\xi f'_{31} - \zeta f'_{11}) + y(\xi f'_{32} - \zeta f'_{12}) + z(\xi f'_{33} - \zeta f'_{13}) = 0 \quad \dots \quad (5)$$

$$x(\eta f'_{31} - \zeta f'_{21}) + y(\eta f'_{33} - \zeta f'_{22}) + z(\eta f'_{33} - \zeta f'_{23}) = 0 \quad \dots \quad (6)$$

Identifying (5) and (6), and eliminating ξ, η, ζ between the relations thus obtained, we obtain the locus of (x', y', z'), which is the Hessian.

Again, eliminating x', y', z' between the same relations, we obtain a relation between ξ, η, ζ, which is the tangential equation of the Steinerian.

Hence we obtain the theorem:

The Steinerian is the envelope of polar lines of all points on the Hessian with respect to the original curve, and it touches the inflexional tangents of the curve.

For a curve of the third order this theorem becomes—

The polar line of a point on the Hessian w.r.t. the cubic touches the Hessian at the double point of the polar conic of the point, i.e., at the corresponding point.

95. That the Steinerian is the envelope of lines two of whose poles coincide may be otherwise shown as follows :—

The first polar of any point (x', y', z') is—

$$x'f_1 + y'f_2 + z'f_3 = 0 \qquad \dots \quad (1)$$

The first polar of any consecutive point—

$$(x' + \delta x', \quad y' + \delta y', \quad z' + \delta z')$$

on the Steinerian is

$$(x' + \delta x')f_1 + (y' + \delta y')f_2 + (z' + \delta z')f_3 = 0$$

$$\therefore \quad \delta x'.f_1 + \delta y'.f_2 + \delta z'f_3 = 0 \qquad \dots \quad (2)$$

From (1) and (2) we obtain—

$$f_1 : f_2 : f_3 = (y'\delta z' - z'\delta y') : (z'\delta x' - x'\delta z') : (x'\delta y' - y'\delta x') \quad \dots \quad (3)$$

$$= \xi : \eta : \zeta \text{ (say)}.$$

i.e., $f_1 : f_2 : f_3$ are proportional to the co-ordinates (ξ, η, ζ) of the tangent to Steinerian. If the two first polars (1) and (2) meet at the point (x'', y'', z''), two poles coincide at this point, which lies on the Hessian, and we have

$$\xi : \eta : \zeta = f''_1 : f''_2 : f''_3 \qquad \dots \quad (4)$$

From (1) it follows, therefore, that, if (x', y', z') are made current, the equation (1) represents the tangent to the Steinerian which, again, in virtue of the relations (4), represents the polar line of the point (x'', y'', z'') on the Hessian. Thus, corresponding to each point (x'', y'', z'') on the Hessian, we obtain a tangent to the Steinerian, and in fact, the polar line of (x'', y'', z'') is identical with the tangent to the Steinerian.

Eliminating (x, y, z) between the equations (3) and the equation of the Hessian, we obtain the tangential equation of the Steinerian in terms of ξ, η, ζ.

96. The Class of the Steinerian :

From what has been said above, the class of the Steinerian can be easily determined. Since the polar lines of points on the Hessian are tangents to the Steinerian, if P is a fixed point on one of these tangents, the first polar of P with respect to the original curve must pass through the coincident poles of the tangent, which lie on the Hessian. Therefore a tangent to the Steinerian, drawn through P, corresponds to a point of intersection of the first polar of P with the Hessian. But the first polar is of order $(n-1)$, and the Hessian of order $3(n-2)$. Therefore they intersect, in general, in $3(n-1)(n-2)$ points, which give as many tangents of the Steinerian drawn from the point P, or, in other words, *the class of the Steinerian is* $3(n-1)(n-2)$.

If, however, the original curve has δ nodes and κ cusps, it may be shown that each node counts as *two*, and each cusp as *four* of the intersections of the first polar and the Hessian. Therefore, the two curves will intersect in only

$$3(n-1)(n-2)-2\delta-4\kappa$$

other points, and consequently the class of the Steinerian is reduced to—

$$3(n-1)(n-2)-2\delta-4\kappa$$

97. A General Theorem :

The class and order of a curve ψ, which is enveloped by the polar lines, with respect to the original n-ic $f=0$, of points on a curve $\phi=0$, of order m, are $m(n-1)$ and $m(2n+m-5)$ respectively.

The class of the envelope in question is determined by the number of tangents which pass through any fixed point (x', y', z'). Since the point lies on the polar line of a point on $\phi=0$, the corresponding point on ϕ must therefore lie

on the first polar of the point (x', y', z'), which is of order $(n-1)$. Therefore, the number of tangents is equal to the number of intersections of $\phi=0$ with the first polar curve of (x', y', z') with respect to the original curve $f=0$,

i.e., with $$x'\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+z'\frac{\partial f}{\partial z}=0 \qquad \cdots \quad (1)$$

These two curves evidently intersect in $m(n-1)$ points, and consequently, the class of the curve ψ is $m(n-1)$.

In order to determine the order of ψ, we take the point (x', y', z') on any line such that two consecutive tangents pass through the point, which therefore lies on the curve ψ. In this case the two points of intersection of $\phi=0$ and (1) will be consecutive points, since to each such point of intersection corresponds one of the tangents. Therefore, the first polars of points on the curve ψ, enveloped by the polar lines of ϕ, touch this latter curve. The condition that the two curves $\phi=0$ and (1) should touch is of degree $m(2n+m-5)$ in the co-efficients of (1), *i.e.*, the condition contains (x', y', z') in the degree $m(2n+m-5)$, which equated to zero gives the locus of points (x', y', z'), whose first polars touch $\phi=0$, *i.e.*, gives the curve $\psi=0$ in point co-ordinates.

Hence, the order of ψ is $m(2n+m-5)$.*

If, however, $\phi=0$ is the Hessian, and $\psi=0$ is the Steinerian, the class and order of ψ are repectively found to be

$$3(n-2)(n-1) \quad \text{and} \quad 3(n-2)\{2n+3(n-2)-5\}$$

i.e., $$3(n-1)(n-2) \quad \text{and} \quad 3(n-2)(5n-11).$$

But, as shown before, the order of the Steinerian is $3(n-2)^2$; this difference is explained by the fact that,

* The locus obtained contains also the inflexional tangents, since at any point on an inflexional tangent, two consecutive tangents meet and consequently such a point will satisfy the condition of the problem, but such tangents are not to be regarded as part of the locus.

when the point (x', y', z') lies on an inflexional tangent, two consecutive tangents through it coincide with the inflexional tangent of ψ. * Therefore, the inflexional tangents of the Steinerian also occur in the equation of the locus obtained, which, however, are not, in general, to be regarded as forming the part of a curve given in line co-ordinates.

Hence, we obtain the theorem:

The locus of points whose first polars touch the Hessian is composed of two parts—the Steinerian of order $3(n-2)^2$, *and another curve of order*

$$3(n-2)(5n-11)-3(n-2)^2=3(n-2)(4n-9)$$

which is the product of the inflexional tangents of the Steinerian.

It should be noted, however, that the first polars of points on the inflexional tangents have proper contact with the Hessian, while the first polars of points on the Steinerian have double points on the Hessian, and consequently there is no contact in the proper sense, although the analytical condition for contact is satisfied at such a point.

Therefore the number of inflexional tangents of the Steinerian is $3(n-2)(4n-9)$.

98. From what has been said above it follows that, if the Steinerian has a double point, that corresponds to two different points on the Hessian, and conversely, if to two different points on the Hessian, there corresponds the same point on the Steinerian, it is a double point on this latter. But, as can be shewn, the Steinerian has $\frac{3}{2}(n-2)(n-3)(3n^2-9n-5)$ double points, and the first polar of each double point has two double points on the Hessian.

* Two tangents are also coincident with a bitangent, but they are not consecutive.

Therefore, there are $\frac{3}{2}(n-2)(n-3)(3n^2-9n-5)$ first polars with two double points, and the corresponding poles are the double points on the Steinerian. The two nodal tangents of the Steinerian are the polar lines of the two corresponding points of the Hessian. It can be proved, in a similar manner, that there are $12(n-2)(n-3)$ first polars which have cusps, the corresponding poles are cusps on the Steinerian and the cuspidal tangents touch the Hessian at those points.*

99. Theorem:

The tangent to the second polar of a point on the Steinerian touches the Hessian at the corresponding point.

Without loss of generality, we may take $A(1,0,0)$ to be a point on the Steinerian and $B(0,1,0)$ the corresponding point on the Hessian. If $f(x, y, z)=0$ is the equation of the curve, the first polar of A is $\frac{\partial f}{\partial x}=0$, and since this has a double point at B, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial x \partial z}$ must vanish at B, *i.e.*, $a=0$, $h=0$, $g=0$ at B, where $a, b, c, \ldots\ldots$ denote the second differential co-efficients of f. The equation of the Hessian is—

$$H \equiv abc+2fgh-af^2-bg^2-ch^2=0$$

The co-ordinates of the tangent to the Hessian at B, since $a=0$, $h=0$, $g=0$, are—

$$\frac{\partial a}{\partial x}\left(bc-f^2\right), \frac{\partial a}{\partial y}\left(bc-f^2\right), \frac{\partial a}{\partial z}\left(bc-f^2\right)$$

and therefore proportional to—

$\frac{\partial a}{\partial x}$, $\frac{\partial a}{\partial y}$, $\frac{\partial a}{\partial z}$, *i.e.*, f_{111}, f_{112}, f_{113} (where $x=0$, $y=1$, $z=0$).

* Clebsch—Leçons sur la Géométrie, Tom II, pp. 87-90.

But the second polar of A is $\dfrac{\partial^2 f}{\partial x^2} = f_{11} = 0.$

$\therefore$ The co-ordinates of the tangent at B to the second polar are the values of $f_{111}, f_{112}, f_{113}$ at the point B.

Hence the Hessian and the second polar of A have a common tangent at B, *i.e., the tangent to the second polar of* A *touches the Hessian at* B.

The equation of the curve in this case may be written as

$$f \equiv ay^n + by^{n-1}z + cy^{n-2}z^2 + y^{n-3}(d_0\ x^3 + 3d_1x^2z +$$
$$+ 3d_2xz^2 + d_3z^3) + y^{n-4}u_4 + \dots\dots\dots = 0$$

where u_r is homogeneous of the rth degree in x and z.

The equation of the common tangent to the second polar and the Hessian at B is $d_0x + d_1z = 0$

Since the second polar of A is the first polar of the first polar of the same point, and the first polar of A has a double point at B, from § 85 we obtain the following theorem :—

The tangent of the Hessian at B *is the harmonic conjugate of the line* BA *with respect to the nodal tangents of the first polar at the point* A.

Again, the tangent at A to the Steinerian is the harmonic conjugate of AB with regard to the two lines constituting the polar conic of B; and the first polar of any point on the tangent touches AB at B.

There is no scope for a detailed discussion of the various properties of the Steinerian and other covariant curves, which have been thoroughly studied by Cremona.*

* Cremona—Introduzione ad una teoria Geometrica delle curve piane. Also Clebsch—Leçons etc., Tom II, Chap. I.

100. The Cayleyan: *

The line joining the point A (x', y', z'), a point on the Steinerian, with the double point B (x'', y'', z'') of its first polar (which is a point on the Hessian) envelopes a third curve, which is called the *Cayleyan* of the original curve. The two points A and B are called the *corresponding* points.

Thus, the Cayleyan may be defined as *the envelope of lines joining points on the Steinerian with their corresponding points on the Hessian.*

The degree of the Cayleyan is $3(n-2)(5n-11)$, and its class is $3(n-1)(n-2)$. It is in fact a contravariant of the original curve. The determination of the order † of the Cayleyan presents some difficulty, and is not consequently attempted here.

101. The Class of the Cayleyan:

Let (x', y', z') and (x'', y'', z'') be two corresponding points on the Steinerian and the Hessian respectively. The line joining them is the tangent to the Cayleyan. If this p asses through a fixed point (ξ, η, ζ), we may take—

$$x' = \lambda\xi + \mu x''$$
$$y' = \lambda\eta + \mu y''$$
$$z' = \lambda\zeta + \mu z''$$

Substituting these values of (x', y', z') in the equation (1) of § 92, we find—

$$\left.\begin{aligned} \mu f''_1 + \lambda(f''_{11}\xi + f''_{12}\eta + f''_{13}\zeta) \equiv \mu f''_1 + \lambda\phi_1 = 0 \\ \mu f''_2 + \lambda(f''_{21}\xi + f''_{22}\eta + f''_{23}\zeta) \equiv \mu f''_2 + \lambda\phi_2 = 0 \\ \mu f''_3 + \lambda(f''_{31}\xi + f''_{32}\eta + f''_{33}\zeta) \equiv \mu f''_3 + \lambda\phi_3 = 0 \end{aligned}\right\} \quad \ldots \quad (1)$$

* Cayley has studied the properties of this curve for a curve of the third order. Phil. Trans., Vol. CXLVII, 2nd part (1857). See also Steiner's papers—Crelle's Journal, Vol. 47. Prof. Cayley himself calls this curve the *Steiner-Hessian*.

† Clebsch—Leçons sur la Géométrie, T. II, pp. 79-82.

whence, eliminating λ, μ, we obtain the equations—

$$\left.\begin{array}{l} f''_2\phi_3-\phi_2 f''_3=0 \\ f''_3\phi_1-\phi_3 f''_1=0 \\ f''_1\phi_2-\phi_1 f''_2=0 \end{array}\right\} \qquad \ldots \quad (2)$$

which give the corresponding values of (x'', y'', z'').

The number of common roots of equations (2) will give the required number of tangents which can be drawn from (ξ, η, ζ) to the curve. But the common roots of the first two are $(2n-3)^2$ in number, from which we must exclude the $(n-1)(n-2)$ common roots of the equations $f_3=0$, $\phi_3=0$, for they satisfy the first two equations and not the third. We must also exclude the case when the point (x', y', z') coincides with (ξ, η, ζ), for that is also a root, but is evidently illusory, as it does not give a definite result. Hence the number of common roots

$$=(2n-3)^2-(n-1)(n-2)-1$$
$$=3(n-1)(n-2)$$

and any of these roots corresponds to a tangent of the Cayleyan passing through (ξ, η, ζ).

Thus *the Class of the Cayleyan is* $3(n-1)(n-2)$.

That the class of the Cayleyan is $3(n-1)(n-2)$ can very easily be shown by means of Chasle's "correspondence formula," which will be explained in a subsequent chapter.

Ex. 1. Prove that the Steinerian and the Cayleyan touch the inflexional tangents.

Ex. 2. Show that the equation of the Cayleyan of the cubic $x^3+y^3+z^3+6mxyz=0$ is $m(\xi^3+\eta^3+\zeta^3)+(1-4m^3)\,\xi\eta\zeta=0$.

Ex. 3. Show that the sides of the triangle of reference constitute the Hessian and Steinerian of the curve—

$$(x/a)^n+(y/b)^n+(z/c)^n=0$$

and the Cayleyan degenerates into the vertices.

Ex. 4. A and B are two corresponding points on the Steinerian and the Hessian respectively. Prove that

(*i*) The tangent at A to the Steinerian is the harmonic conjugate of AB for the degenerate polar conic of B.

(*ii*) If AB touches the Hessian at B, it touches the Cayleyan at B.

102. *The Hessian passss throuyh all the double points on the curve.*

We have seen (§ 72) that the polar conic of a double point breaks up into two right lines. But the Hessian is the locus of points whose polar conics are two right lines. Consequently the double point must be a point on the Hessian.

Or, we may proceed directly as follows:—

If (x', y', z') is a double point on the curve $f=0$, we have

$$f'_1=f'_2=f'_3=0.$$

But, by Euler's theorem on homogeneous functions,

$$x'f'_{11}+y'f'_{12}+z'f'_{13}=(n-1)f_1'=0$$
$$x'f'_{21}+y'f'_{22}+z'f'_{23}=(n-1)f_2'=0$$
$$x'f'_{31}+y'f'_{32}+z'f'_{33}=(n-1)f_3'=0$$

Eliminating (x', y', z') between these equations, we obtain

$$H\equiv\begin{vmatrix} f'_{11} & f'_{12} & f'_{13} \\ f'_{21} & f'_{22} & f'_{23} \\ f'_{31} & f'_{32} & f'_{33} \end{vmatrix}=0$$

which shows that (x', y', z') is a point on the Hessian.

103. *Every node on a curve is a node on the Hessian with the same nodal tangents.*

The equation of a curve, having the origin as a node with the axes of x and y as nodal tangents, may be written in the homogeneous form as—

$$F\equiv xyz^{n-2}+u_3z^{n-3}+\ldots+u_n=0. \qquad \ldots \quad (1)$$

Now, $$\frac{\partial F}{\partial x}=yz^{n-2}+\frac{\partial u_3}{\partial x}\ z^{n-3}+\ldots$$

$$\frac{\partial F}{\partial y}=xz^{n-2}+\frac{\partial u_3}{\partial y}\ z^{n-3}+\ldots$$

$$\frac{\partial F}{\partial z}=(n-2)xyz^{n-3}+(n-3)u_3z^{n-4}+\ldots$$

Therefore, we have—

$$a\equiv\frac{\partial^2 F}{\partial x^2}=\frac{\partial^2 u_3}{\partial x^2}\ z^{n-3}+\frac{\partial^2 u_4}{\partial x^2}\ z^{n-4}+\ldots$$

$$b\equiv\frac{\partial^2 F}{\partial y^2}=\frac{\partial^2 u_3}{\partial y^2}\ z^{n-3}+\ldots$$

$$c\equiv\frac{\partial^2 F}{\partial z^2}=(n-2)(n-3)xyz^{n-4}+\ldots$$

$$f\equiv\frac{\partial^2 F}{\partial y\partial z}=(n-2)xz^{n-3}+\ldots$$

$$g\equiv\frac{\partial^2 F}{\partial z\partial x}=(n-2)yz^{n-3}+(n-3)\frac{\partial u_3}{\partial x}\ z^{n-4}+\ldots$$

$$h\equiv\frac{\partial^2 F}{\partial x\partial y}=z^{n-2}+\frac{2^2 u_3}{\partial x\partial y}\ z^{n-3}+\ldots$$

The equation of the Hessian may be symbolically written as—

$$abc+2fgh-af^2-bg^2-ch^2=0 \qquad \ldots \quad (2)$$

and it will be seen that lowest terms in this are of order two. For the orders of the lowest terms in the second differential co-efficients a, b, c, etc., are respectively 1, 1, 2, 1, 1, 0.

∴ The order of the lowest terms in abc is $1+1+2=4$

,,	,,	,,	fgh is	$1+1+0=2$
,,	,,	,,	af^2 is	$1+2=3$
,,	,,	,,	bg^2 is	$1+2=3$
,,	,,	,,	ch^2 is	$2+0=2$

Thus it is seen that the order in the variables of the lowest terms in the equation (2) of the Hessian is two, and the origin is consequently a double point on the Hessian.

Again, the lowest terms in (2) occur only in fgh and ch^2, each of which contains xy as a factor.

Therefore the origin is a node on the Hessian with the axes as tangents, which are also the nodal tangents to the given curve.

It is to be noted that a node on a curve counts as six of its intersections with the Hessian. (§ 51.)

Ex. Consider the curve $x^3+y^3=3axyz$ (Folium of Descartes).

The Hessian is $H\equiv x^3+y^3+axyz=0$, which is another curve of the same form, whose loop lies on the opposite side. Both the curves have a node at the origin with the axes of x and y as tangents.

104. *Every cusp on a curve is a triple point on the Hessian, and two of the tangents at the triple point coincide with the cuspidal tangent.*

The equation of a curve having the origin for a cusp with the axis of y as the cuspidal tangent is—

$$F\equiv x^2z^{n-2}+u_3z^{n-3}+\ \dots\ +u_n=0$$

Now, the second differential co-efficients are respectively

$$a=2z^{n-2}+\dots \qquad b=\frac{\partial^2 u_3}{\partial y^2}\,z^{n-3}+\dots$$

$$c=(n-2)(n-3)x^2z^{n-4}+\dots$$

$$f=(n-3)\frac{\partial u_3}{\partial y}\,z^{n-4}+\dots \qquad g=2(n-2)xz^{n-3}+\dots$$

$$h=\frac{\partial^2 u_3}{\partial x\partial y}z^{n-3}+\dots$$

The orders of the lowest terms in the second differential co-efficients a, b, c, etc., are therefore 0, 1, 2, 2, 1, 1, respectively.

Hence it is seen that the order of the lowest terms in the equation of the Hessian $abc+2fgh-af^2-bg^2-ch^2=0$

is *three*. Thus the equation of the Hessian begins with the third order terms, and consequently the origin is a *triple point* on the Hessian.

Again, the lowest terms occur only in abc and bg^2, and each of these contains x^2 as a factor. Hence the cuspidal tangent occurs also as two coincident tangents to the Hessian at the triple point. Thus the triple point on the Hessian arises from a cusp with a simple branch passing through it, and two of the tangents at the triple point on the Hessian coincide with the cuspidal tangent of the original curve.

It is to be noted that a cusp on a curve counts as eight of its intersections with the Hessian (§51).

Ex. Consider the curve $x(x^2+y^2)=2ky^2$ (the Cissiod).

The origin is a cusp, $y=0$ being the cuspidal tangent. The Hessian is $xy^2=0$, which has a triple point at the origin, two tangents at which are coincident with the cuspidal tangent $y=0$.

105. *A multiple point of order k on a curve is a multiple point of order $3k-4$ on the Hessian, and the tangents at the multiple point are tangents to the Hessian.*

The equation of a curve, having the origin for a multiple point of order k, with the axis of y for one of the tangents, is

$$\mathrm{F} \equiv xu_{k-1}z^{n-k} + u_{k+1}z^{n-k-1} + \ldots\ldots + u_n = 0.$$

The degree of the second differential co-efficients may be determined as follows:—

Where there are two differentiations with respect to x or y, the degree of the lowest terms will be $k-2$; where there is one differentiation with respect to z and one with respect to x or y, the degree is $k-1$; and where both differentiations are performed with respect to z, the degree is k. Thus the degree of the lowest terms will be $k-2$ in a, $k-2$ in b, k in c, $k-1$ in f, $k-1$ in g, and $k-2$ in h.

Hence the degree of the lowest terms in the equation of the Hessian $abc+2fgh-af^2-bg^2-ch^2=0$ will be $3k-4$.

Therefore the origin is a *multiple point* of order $3k-4$ on the Hessian.

Further, the lowest terms in the equation of the curve contain x as a factor. Then it is evident that x will be a factor in the lowest terms of each of the second differentials in which no differentiation has been performed with respect to x, *i.e.*, x will be a factor in b, c and f. But every term of $abc+2fgh-af^2-bg^2-ch^2$ contains either b, c or f. Therefore the lowest terms in the equation of the Hessian contain x as a factor, and consequently, $x=0$ is a tangent to the Hessian. Similarly, it can be shown that all the tangents at the multiple point on the curve are tangents to the Hessian.*

106. Harmonic Polar:

We have seen (§71) that the polar conic of the origin is $n(n-1)u_0+2(n-1)u_1+2u_2=0$. If the origin is a point of inflexion on the curve, we must have—

$$u_0=0,\ u_2=u_1v_1$$

Therefore the equation of the polar conic becomes—

$$\{(n-1)+v_1\}u_1=0$$

i.e., the polar conic breaks up into two right lines, one of which is the tangent $u_1=0$, and the other is the line $(n-1)+v_1=0$, which is called the *Harmonic Polar* of the point of inflexion. It follows, therefore, that the origin, which is a point of inflexion on the given curve, is a point on the Hessian.

* A multiple point of order k on the curve is a multiple point of order $(3k-4)$ on the Hessian, and moreover, the two curves have k tangents common. Therefore, the point counts as $k(3k-4)+k$, or $3k(k-1)$ intersections (§ 51). But since $3k(k-1)=6.\dfrac{k(k-1)}{2}$, and a node on the curve counts as 6 intersections with its Hessian, the multiple point may be regarded as resulting from the union of $\dfrac{k(k-1)}{2}$ nodes, a result we have otherwise obtained in §46.

107. Number of Points of Inflexion:

The points of inflexion on a curve are the points of intersection of the curve with its Hessian, and their number is $3n(n-2)$, when the curve has no singular points.

Let us examine the conditions when the line joining two given points (x, y, z) and (x', y', z') intersects the curve in three consecutive points. The co-ordinates of any point on the line are $\lambda x+\mu x'$, $\lambda y+\mu y'$, $\lambda z+\mu z'$. Substituting these values for x, y, z in the equation $f(x, y, z)=0$ of the curve, we obtain the equation (1) or (2) of § 63 to determine the ratio $\lambda:\mu$. If the point (x', y', z') lies on the curve, $f'=0$, and one value of $\lambda:\mu$ is zero. If further $\Delta' f'=0$ and $\Delta'^2 f'=0$, two other roots will be zero, and the line will meet the curve in three coincident points at (x', y', z'). Therefore, in this case (x', y', z'), which is a variable point, satisfies both $\Delta' f'=0$, $\Delta'^2 f'=0$.

The first condition shows that the point lies on the line $\Delta' f'=0$, *i.e.*, on the tangent at (x', y', z'). The second condition requires that it lies on the polar conic of (x', y', z'). Combining these two conditions, it follows that the polar conic breaks up into two right lines, one of which is the tangent at the point. Hence the point (x', y', z') is a point on the Hessian, or, what is the same thing that—

The Hessian passes through the points of inflexion on a curve.

Now, the degree of the Hessian is $3(n-2)$, and therefore it intersects the curve of the nth degree in $3n(n-2)$ points. If the curve has no other singularities, this must exactly be the number of points of inflexion on the curve. If, however, the curve possesses other singularities, this number must be reduced.

Ex. Show that the abscissae of the points of inflexion on the curve $y^n=f(x)$ are the roots of the equation

$$\frac{n-1}{n}\{f'(x)\}^2=f(x)\, f''(x).$$

108. Discrimination of a double Point from a Point of Inflexion:

We have seen that the polar conics of both a double point and a point of inflexion break up into right lines, and consequently both must lie on the Hessian. Therefore, it is necessary to devise means of discriminating whether a point on a curve is a double point or a point of inflexion. There are various ways of doing this:—

(a) If we transfer the origin to the point in question, the lowest degree terms in the transformed equation will be *linear*, if the origin is a point of inflexion; and further in this case, the terms of the second degree will contain the linear terms as a factor. But if it is a double point, the linear terms will be absent from the equation, and the lowest terms will be a quadratic.

(b) The polar conic of a double point breaks up into two right lines—the nodal tangents, which are also nodal tangents to the Hessian. But the polar conic of a point of inflexion breaks up into two right lines, one of which is the tangent at the point, and the other does not, in general, pass through the point, and the point is a single point on the Hessian.

(c) At a double point (x', y', z'), we have—

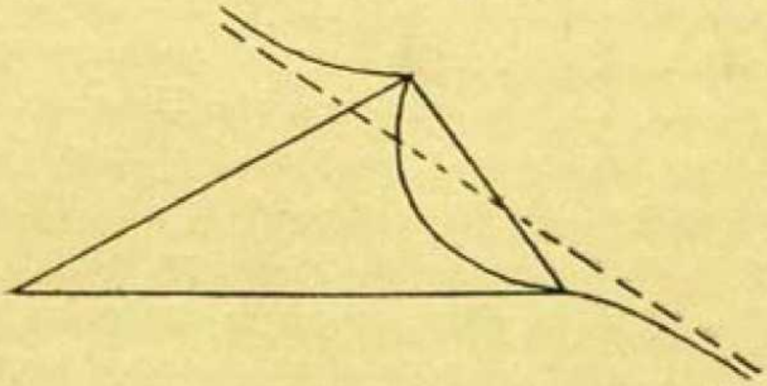
$$\frac{\partial f'}{\partial x'}=0, \qquad \frac{\partial f'}{\partial y'}=0, \qquad \frac{\partial f'}{\partial z'}=0$$

while at a point of inflexion these functions do not *all* vanish, but have definite values, and are proportional to the co-ordinates of the tangent at the point.

Ex. Consider the curve $x^3=y^2z$.

This is evidently a cubic, having the vertex $C(0,0,1)$ as a cusp, with $y=0$ as the cuspidal tangent, while the vertex B is a point of inflexion, with the inflexional tangent $z=0$. The Hessian is evidently $xy^2=0$, which passes through A, B, C, and therefore intersects the

cubic at B and C. Hence B and C may be either double points or points of inflexion. To determine this, we see that the first differential co-efficients *all* vanish at C, and therefore it is a double point,



evidently a cusp, with the cuspidal tangent $y=0$, common to the curve and the Hessian. At the point B, however,

$$\frac{\partial f}{\partial x}=0, \quad \frac{\partial f}{\partial y}=-2yz=0,$$

$$\frac{\partial f}{\partial z}=-y^2=-1,$$

i.e., these functions do not *all* vanish at B. Therefore the point B is a point of inflexion, with $z=0$ as the inflexional tangent. The form of the curve is shown in the accompanying figure.

109. A Theorem :

On the polar line of any point A, there are always three different points, of which the first polars have points of inflexion at A.

The polar line of a point A (x', y', z') is—

$$x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'}+z\frac{\partial f}{\partial z'}=0 \qquad \dots \quad (1)$$

Let B (x'', y'', z'') be a point on (1).

The first polar of B is—

$$F\equiv x''\frac{\partial f}{\partial x}+y''\frac{\partial f}{\partial y}+z''\frac{\partial f}{\partial z}=0 \qquad \dots \quad (2)$$

and it passes through A (§ 76).

The Hessian of F is

$$\begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} = 0 \qquad \ldots \quad (3)$$

The equation (3) involves (x'', y'', z'') in the third order.

Now, if A (x', y', z') is a point of inflexion on (2), (x', y', z') must satisfy (3). Hence, if we consider (x', y', z') as given, the locus of (x'', y'', z'') is a curve of the third order given by (3). Also (x'', y'', z'') is a point on the line (1). Therefore (x'', y'', z'') is any one of the three points of intersection of the curve (3) with the line (1).

110. If, however, the point B lies on a curve $\phi=0$, the points of inflexion of its first polar lie on a curve, whose equation is obtained by eliminating (x'', y'', z'') between the equations (2), (3) and

$$\phi(x'', y'', z'')=0.$$

If, in particular, $\phi=0$ is a straight line

$$\xi x+\eta y+\zeta z=0 \qquad \ldots \quad (4)$$

the eliminant is an equation $\Theta=0 \qquad \ldots \quad (5)$

of order $6(n-2)$. Since (2) and (4) are each linear and (3) is of degree 3 in (x'', y'', z''), Θ is of degree 3 in the co-efficients of (2), of degree 1 in the co-efficients of (3), and of degree 3 in those of (4). But the co-efficients of (2), (3) and (4) are of orders $n-1$, $3n-9$ and 0, respectively, whence the order of Θ in the variables is

$$3(n-1)+(3n-9)+0=6(n-2).$$

The first polars of all points on the line pass through $(n-1)^2$ common points, which are the poles of the line. But these $(n-1)^2$ points are *triple* points on the curve

$\Theta=0$. For from (2) and (4), the $(n-1)^2$ poles of the line are given by

$$f_1 : f_2 : f_3 \equiv \xi : \eta : \zeta.$$

When these are satisfied, the second partial differential co-efficients of Θ *w.r.t.* (x, y, z) vanish identically, *i.e.*, $\Delta^2\Theta \equiv 0$, which shows that (x, y, z) is a triple polnt on $\Theta=0$.*

If, on the other hand, (x, y, z) are regarded as constants in the equation $\Theta=0$, and ξ, η, ζ as variables, the equation gives the product of the three linear factors, which equated to zero give the equations of the three poles, whose first polars have a point of inflexion at (x, y, z).

If the line envelopes a curve $\phi(\xi, \eta, \zeta)=0$, of class m, the $(n-1)^2$ points of intersection of the first polars describe the curve $\phi(f_1, f_2, f_3)=0$, of order $m(n-1)$. If, in particular, the line turns about a point (α, β, γ), the curve described is the first polar of that point.

111. *If a curve has δ nodes, the number of its points of inflexion cannot exceed $3n(n-2)-6\delta$.*

The Hessian of a curve passes through all the double points and points of inflexion on a curve, and it intersects the curve in $3n(n-2)$ points, which include all these singularities. But a node on a curve counts as six among the intersections of the curve with its Hessian (§ 103), and therefore the δ nodes are equivalent to 6δ intersections. Thus, the remaining $3n(n-2)-6\delta$ intersections give only the points of inflexion.

112. *If a curve has κ cusps, the number of its points of inflexion cannot exceed $3n(n-2)-8\kappa$.*

Since each cusp counts as eight among the intersections of a curve with its Hessian (§ 104), the κ cusps are equivalent to 8κ intersections, and the Hessian intersects

*Clebsch, Leçons, Tom II, p. 20.

the curve only in $3n(n-2)-8\kappa$ other points, which are the points of inflexion on the curve.

Combining this with the result of the preceding article, we obtain the theorem :

If a curve has δ nodes and κ cusps, the number of its points of inflexion cannot exceed $3n(n-2)-6\delta-8\kappa$.

Ex. 1. Form the Hessian of the curve $x^2(x^2+y^2)=a^2y^2$, and find the points of inflexion.

$$[y^2(2x^4-x^2y^2+6a^2x^2+a^2y^2)=0]$$

Ex. 2. Show that the curve $axy+a^3=x^3$ has a point of inflexion at the point where it cuts the axis of x, and show that the tangent at the point of inflexion is inclined to the axis of x at an angle $\tan^{-1} 3$.

$$[\text{Hessian} \quad 3x^3-3a^3+axy=0.]$$

Tangent at $(a, 0)$ is $y=3x$, when origin is transferred to the point $(a, 0)$.

Ex. 3. Find the inflexions on the curve—

$$\frac{y}{c}=\frac{x^2}{9a^2}+\left(\frac{x-a}{a}\right)^{\frac{1}{3}}$$

Ex. 4. Show that the inflexions on the cubic $(x^2+a^2)y=a^2x$ are given by $x=0$ and $x=\pm a\sqrt{3}$.

Ex. 5. Find the points of inflexion on the curves—

(*i*) $2x(x^2+y^2)=a(2x^2+y^2)$ (*ii*) $x^3+y^3=a^3$.

(*iii*) $x^3+(a-y)x^2+a^2y+a^3=0$.

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CHAPTER VI

POLAR RECIPROCAL AND OTHER DERIVED CURVES.

113. Point Reciprocation :

In Chapter I, we have enunciated the general principles of reciprocation.* The theory of point reciprocation has been fully discussed by Salmon in his Treatise on Conic Sections, Chapter XV, and therefore it is not necessary to enter into a detailed discussion of the theory in the present work. We shall here study the properties of polar reciprocal curves mainly with reference to their singular points. At the outset, however, it is useful to recall the following definition :

Suppose we have a certain conic, called the base-conic. The locus of the pole, *w.r.t.* the base-conic, of any tangent to the given curve **S** is called the polar reciprocal **S′** of **S** with regard to the conic. The relation between **S** and **S′** is reciprocal, one being derived from the other by the same process.

Let $f=0$ be the base-conic, and $\mathbf{S}=0$ any given curve.

Let the equation of a tangent to **S** be written in the form

$$p=x\cos\alpha+y\sin\alpha \qquad \ldots \quad (1)$$

and the condition that this line touches the given curve be written in the form

$$p=\phi(\alpha) \qquad \ldots \quad (2)$$

If now (x', y') be the pole of this tangent *w.r.t.* $f=0$, the tangent (1) must be identical with the polar line of (x', y'), *i.e.*, with

$$xf'_x+yf'_y+zf'_z=0 \qquad \ldots \quad (3)$$

$$\therefore \quad \frac{\cos\alpha}{p}=\frac{-f'_x}{f'_z} \quad \text{and} \quad \frac{\sin\alpha}{p}=\frac{-f'_y}{f'_z}$$

* Poncelet, Memoire sur la theorie generale des polaires reciproques, etc. Crelle, Bd. IV. (1829), pp. 1-71.

$$\frac{1}{p^2} = \frac{f'^2_x + f'^2_y}{f'^2_z}, \quad \text{and} \quad \tan \alpha = \frac{f'_y}{f'_x}.$$

Therefore, the locus of (x', y'), *i.e.*, the polar reciprocal of **S** is

$$f^2_x + f^2_y = f^2_z \left\{ \phi \left(\tan^{-1} \frac{f_y}{f_x} \right) \right\}^2$$

The base $f = 0$ may be any conic—a circle for example, and a point-circle, in particular ; and in that case, the reciprocal is said to be taken *w.r.t.* the circle.

In this case we may define the polar reciprocal curve in a different manner as follows :

If OP be the perpendicular from the pole upon the tangent to a given curve, and if on OP, or OP produced, a point Q be taken such that OP. OQ $= k^2$ (*const.*), the locus of Q is called the *reciprocal polar* of the given curve with regard to a circle of radius k and centre at O.

It is evident from this definition that the polar reciprocal curve is the inverse of the first positive pedal of the given curve.

Ex. 1. Find the polar reciprocal of the curve

$$(x/a)^m + (y/b)^m = 1$$

with regard to a circle of radius k and centre at the origin.

The condition that the line $x \cos \alpha + y \sin \alpha = p$ is a tangent to the curve is—

$$(a \cos \alpha)^{\frac{m}{m-1}} + (b \sin \alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

If now r denotes the radius vector of the reciprocal polar, we have $pr = k^2$, and substituting the value of p in the above condition, we obtain

$$(a \cos \alpha)^{\frac{m}{m-1}} + (b \sin \alpha)^{\frac{m}{m-1}} = (k^2/r)^{\frac{m}{m-1}}$$

whence

$$(ax)^{\frac{m}{m-1}} + (by)^{\frac{m}{m-1}} = k^{\frac{2m}{m-1}}.$$

If, in particular, $m=2$, the polar reciprocal of the ellipse

$$x^2/a^2+y^2/b^2=1$$

is found to be $a^2x^2+b^2y^2=k^4$.

Ex. 2. Show that the polar reciprocal of the curve $x^m y^n = a^{m+n}$ with regard to a circle whose centre is at the origin is another curve of the same kind.

Ex. 3. Show that the polar reciprocal of the curve $r^m = a^m \cos m\theta$ *w.r.t.* the hyperbola $r^2 \cos 2\theta = a^2$ is—

$$r^{\frac{m}{m+1}} \cos\left(\frac{m}{m+1}\right)\theta = a^{\frac{m}{m+1}} \cdot$$

Ex. 4. Prove that the reciprocal of the same curve *w.r.t.* a circle with centre at the pole is of the form in Ex. 3.

114. Polar Reciprocal in Homogeneous Co-ordinates:

Theorem. *If* $\phi\,(\xi, \eta, \zeta)=0$ *be the tangential equation of a curve, the point equation of its reciprocal polar with respect to the imaginary circle* $x^2+y^2+z^2=0$ *is*

$$\phi(x, y, z)=0.$$

Let $\xi x+\eta y+\zeta z=0$ be a tangent to the curve, and let (x', y', z') be its pole with respect to $x^2+y^2+z^2=0$. Therefore, the polar of (x', y', z'), *i.e.*, $xx'+yy'+zz'=0$ must be identical with $\xi x+\eta y+\zeta z=0$.

i.e., we must have

$$\frac{\xi}{x'}=\frac{\eta}{y'}=\frac{\zeta}{z'} \;;$$

and consequently, $\phi(x', y', z')=0$, *i.e.*, the locus of the point (x', y', z') is the curve $\phi(x, y, z)=0$, which is the reciprocal polar of the given curve.

Thus, if m be the degree of the tangential equation of a curve, the degree of its reciprocal polar curve is also m. But the degree of the tangential equation of a curve is equal to its class.*

* Salmon, Conics, § 321.

Therefore, *the degree of the reciprocal polar of a curve is equal to its class*, and *vice versa*.

The conic $x^2+y^2+z^2=0$ is called the base-conic.

It is to be noticed, in particular, that if the condition that the line $\xi x+y+\zeta=0$ touches a curve be put into the form $\phi(\xi, \zeta)=0$, *i.e.*, if the tangential equation of the curve be $\phi(\xi, \zeta)=0$, then its polar reciprocal with regard to the parabola $x^2+2y=0$ is $\phi(x, y)=0$.

Ex. 1. Find the reciprocal polar of the curve $z(z^2-xy)=xy(x+y)$ *w. r. t.* $x^2+y^2+z^2=0$.

Ex 2. Prove that the polar reciprocal of $\phi(\xi, \eta, \zeta)=0$ with respect to the base-conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

is
$$\phi(ax+hy+gz, \quad hx+by+fz, \quad gx+fy+cz)=0.$$

Ex. 3. Show that a conic and its reciprocal have, in general, a common self-conjugate triangle with the base-conic.

115. Tangential Equation derived from Point-Equation.

Let $f(x, y, z)=0$ be the equation of a curve, and let $\xi x+\eta y+\zeta z=0$ be a tangent to it at the point (x', y', z'). Then this line must be identical with

$$x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'}+z\frac{\partial f}{\partial z'}=0,$$

which is the equation of the tangent at (x', y', z').

$\therefore$ We have $\xi : \eta : \zeta = f'_1 : f'_2 : f'_3 \qquad \dots \quad (1)$

Since (x', y', z') is a point on $f(x, y, z)=0$, we have

$$f(x', y', z')=0 \qquad \dots \quad (2)$$

Eliminating (x', y', z') between equations (1) and (2), we obtain the required tangential equation of the curve.

But, by Euler's Theorem,

$$x' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y'} + z' \frac{\partial f}{\partial z'} = nf(x', y', z') = 0 \quad \ldots \quad (3)$$

Therefore, the process is simplified, if elimination is effected between (1) and (3). We thus obtain the tangential equation in the form $\phi(\xi, \eta, \zeta) = 0$.

Second Method :

Eliminating one of the variables z (say) between the equation of the curve

$$f(x, y, z) = 0 \quad \ldots \quad (1)$$

and

$$\xi x + \eta y + \zeta z = 0 \quad \ldots \quad (2)$$

we obtain the equation of the lines—

$$f\,(x, y, -(\xi x + \eta y)/\zeta) = 0 \quad \ldots \quad (3)$$

which join the vertex C(0, 0, 1) of the triangle of reference with the intersections of (1) with (2).

If the line (2) touches the curve (1), two of the lines (3) will be coincident, and the equation (3), regarded as an equation in x/y, will have two equal roots. Hence the discriminant of (3) will vanish, and this gives the tangential equation of the curve.

Ex. 1. Find the tangential equation of the cubic $x^3 + y^3 + z^3 = 0$.

First Method :—

We have— $\xi : \eta : \zeta = x'^2 : y'^2 : z'^2$

and also $\xi x' + \eta y' + \zeta z' = 0$.

But, $x' = \pm \sqrt{\xi}$, $y' = \pm \sqrt{\eta}$, $z' = \pm \sqrt{\zeta}$.

$$\therefore \pm \xi^{\frac{3}{2}} \pm \eta^{\frac{3}{2}} \pm \zeta^{\frac{3}{2}} = 0;$$

or, rationalising, we obtain—

$$\varphi \equiv \zeta^6 + \eta^6 + \zeta^6 - 2\zeta^3\eta^3 - 2\xi^3\zeta^3 - 2\xi^3\eta^3 = 0$$

which is the tangential equation required.

Second Method :—

Eliminating z between the two equations, we obtain—

$$x^3(\xi^3-\zeta^3)+3\xi^2\eta x^2y+3\xi\eta^2xy^2+(\eta^3-\zeta^3)y^3=0.$$

The discriminant of this is—

$$\zeta^6(\xi^3+\eta^3-\zeta^3)^2=4\xi^3\eta^3\zeta^6.$$

But, $\zeta=0$ does not make the line a tangent, and consequently, the factor ξ^6 is irrelevant. Hence the tangential equation is—

$$(\xi^3+\eta^3-\zeta^3)^2=4\xi^3\eta^3$$

which is the same equation as above.

Ex. 2. The tangential equation of the parabola $y^2=4ax$ is—

$$a\eta^2=\xi\zeta.$$

Ex. 3. The tangential equation of $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}z^{\frac{2}{3}}$ is found to be

$$(\xi^2+\eta^2)\zeta^2=a^2\xi^2\eta^2.$$

Ex. 4. Find the tangential equations of the following curves :—

(*i*) $3(x+y)z^2=x^3$ (*ii*) $x(x^2+y^2)=y^3$.

(*iii*) $(x^2+y^2)^5=a(3x^2y-y^3)$.

Ex. 5. Find the tangential equation of the circular points.

116. The Point-Equation derived from Tangential Equation :

Let $\phi(\xi, \eta, \zeta)=0$ be the given tangential equation of a curve, and $\xi x+\eta y+\zeta z=0$ be a tangent to the curve drawn from any point (x, y, z).

If ζ be eliminated between these two equations, the resulting equation will determine the co-ordinates of the tangents which can be drawn from (x, y, z) to the curve. Now, if the point lies on the curve, two of these tangents must coincide, *i.e.*, the resulting equation should have a pair of equal roots.

Consequently, the discriminant $f(x, y, z)$ must vanish, which gives $f(x, y, z)=0$ as the point equation of the curve.

Second Method :—

Let (x, y, z) be a point, and ξ', η', ζ' be the co-ordinates of its polar line. But the pole of the line (ξ', η', ζ') is—

$$\xi \frac{\partial \phi}{\partial \xi'} + \eta \frac{\partial \phi}{\partial \eta'} + \zeta \frac{\partial \phi}{\partial \zeta'} = 0 \text{ *}$$

$$\therefore \quad x : y : z = \frac{\partial \phi}{\partial \xi'} : \frac{\partial \phi}{\partial \eta'} : \frac{\partial \phi}{\partial \zeta'}$$

$$\text{Also,} \quad \xi' \frac{\partial \phi}{\partial \xi'} + \eta' \frac{\partial \phi}{\partial \eta'} + \zeta' \frac{\partial \phi}{\partial \zeta'} = m\phi(\xi', \eta', \zeta') = 0,$$

where m is the degree of ϕ, since the line (ξ', η', ζ') touches the curve.

Eliminating (ξ', η', ζ) between these equations, we obtain the locus of (x, y, z) in the form $f(x, y, z) = 0$.

Ex. 1. Form the point-equation of the curve whose tangential equation is $\xi^3 + \eta^3 + \zeta^3 = 0$.

Eliminating ζ between this equation and $\xi x + \eta y + \zeta z = 0$, we obtain

$$\xi^3(x^3 - z^3) + 3\xi^2 \eta x^2 y + 3\xi \eta^2 x y^2 + (y^3 - z^3)\eta^3 = 0.$$

The discriminant of this is—

$$z^6(x^3 + y^3 - z^3)^2 = 4x^3 y^3 z^6.$$

But, evidently $\xi x + \eta y + \zeta z = 0$ is not a point on the curve, if $z = 0$.

$\therefore$ The factor z^6 is irrelevant, and the point-equation of the curve is

$$x^6 + y^6 + z^6 - 2x^3y^3 - 2y^3z^3 - 2z^3x^3 = 0.$$

Thus the order of this curve, which is of class 3, is 6.

Ex. 2. The point-equation of $4\xi^3 + 27\eta^2\zeta = 0$ is found to be

$$x^3 - y^2 z = 0$$

* C. A. Scott, *loc. cit.*, § 64.

For, eliminating ζ between the given equation and $\xi x+\eta y+\zeta z=0$, we obtain—

$$4\xi^3 z-27\eta^2\xi x-27\eta^3 y=0$$

the discriminant of which is x^3-y^2z. Hence the point-equation is $x^3=y^2z$, and the order of this curve, which is of class 3, is also 3.

Ex. 3. Find the point-equation of curves defined by the following tangential equations :—

$$(i)\ 9\eta+4(\xi-\eta)^3=0. \qquad (ii)\ (\xi^2+\eta^2)\zeta^2=a^2\,\xi^2\,\eta^2.$$

$$(iii)\ 27a^2\eta^2\zeta+4(a\xi+\zeta)^3=0.$$

117. Principle of Duality :

By a comparison of the examples in §§ 115, 116, it is seen that the class of a cubic may be as much as 6, or as little as 3; and conversely, the degree of a curve of class 3 may be either 6 or 3. Hence, in general, it follows that the class of a curve does not depend solely on its order, nor its order upon the class. Further, it is seen that if $f(x,\ y,\ z)=0$ is the point-equation of a curve whose tangential equation is $\phi(\xi,\ \eta,\ \zeta)=0$, then $f(\xi,\ \eta,\ \zeta)=0$ is the tangential equation of a curve whose point-equation is $\phi(x,\ y,\ z)=0$, and the class of a curve is the same as the degree of its tangential equation.

Now, corresponding to the locus $f(x,\ y,\ z)=0$, there is, by the Principle of Duality,* an envelope $f(\xi,\ \eta,\ \zeta)=0$, and the point-equation of this, found by the above method, is $\phi(x,\ y,\ z)=0$. Instead of discussing the two equations $f(x,\ y,\ z)=0$ and $\phi(\xi,\ \eta,\ \zeta)=0$ of the same curve, we may conveniently consider two distinct curves $f(x,\ y,\ z)=0$ and $\phi(x,\ y,\ z)=0$. These two curves are called *Polar Reciprocal* curves. The relation between them is a reciprocal one, *i.e.*, the one can be obtained from the other in the same way as the latter is from the former; and the line and point properties of one are exactly the same as the point and line properties of the other.

* Plücker, " System der Analytischen Geometrie (1835).

118. Theorem:

A node on a curve corresponds to a bitangent on the reciprocal curve, and *vice versâ*.

Through a node there pass two branches of the curve, with a distinct tangent to each. Thus the point may be regarded once as belonging to one branch and once to the other. Therefore, the polar lines of these two points are two corresponding tangents to the reciprocal curve, which become ultimately coincident. The nodal tangents correspond to two distinct points on the reciprocal, the tangents at which become ultimately coincident. Thus, the nodal tangents with their coincident points of contact reciprocate into two distinct points on the reciprocal curve, with the tangents at those points coincident. Hence, a node corresponds to a tangent of the reciprocal curve, which touches it at two distinct points, *i.e.*, is a bitangent.

Note. Since a conjugate point is a real point with imaginary tangents, the polar line of the conjugate point is a real tangent with imaginary points of contact.

Hence, *a conjugate point on a curve corresponds to a real double or bitangent of the reciprocal curve with imaginary points of contact.*

119. Theorem:

A cusp on a curve corresponds to an inflexional tangent on the reciprocal curve and *vice versâ*.

At a cusp, the two branches of the curve have a common tangent. Let P be the cusp, and PQ the cuspidal tangent. From any point Q on the tangent, we can draw two other tangents QT, QR, one to each branch of the curve. The reciprocals of these three concurrent tangents will be three points on the reciprocal curve, which are collinear on the polar of Q. Ultimately when Q moves up to coincidence with P, the three tangents coincide with the cuspidal tangent, and the three corresponding points become collinear, *i.e.*, three consecutive

points on the reciprocal curve are collinear. Therefore the tangent has a contact of the second order, and consequently, the cusp corresponds to an inflexional tangent on the reciprocal curve.

Second Proof:

At a cusp the moving point turns back along the tangent, *i.e.*, it changes its sense of motion in the direction of the tangent. Therefore, in the reciprocal curve, the enveloping line changes its direction of motion about its point of contact, *i.e.*, the line becomes stationary for a moment before the sense of its motion is changed. Therefore it is an inflexional tangent, and the point of contact corresponds to the cuspidal tangent.

Thus we see that a cusp and an inflexional tangent are stationary elements, while the node and the double tangent are simply double elements.*

120. Theorem:

If a curve of order n has δ nodes, the degree of its reciprocal polar is $n(n-1)-2\delta$.

We have seen that the reciprocal of a curve of order n is, in general, of degree $n(n-1)$, this being the number of tangents that can be drawn from any point to the curve, and this is equal to the number of intersections of the curve with its first polar. If, however, the curve has a node, the first polar passes through it, and the point counts as two among the intersections of the curve with its first polar. But the line joining the node to the pole is not to be reckoned as a tangent. Thus the number of tangents which can be drawn from any point to the

* At a tacnode, two distinct branches of a curve touch, and have a common tangent and a common point. Therefore, in the reciprocal curve there are two branches having a common tangent at a common point, *i.e.*, *to a tacnode corresponds a tacnode on the reciprocal curve.*

curve is diminished by *two* for every node on the curve. Therefore, if there are δ nodes, the number of tangents is diminished by 2δ, and consequently this number reduces to $n(n-1)-2\delta$; or, in other words—

The degree of the reciprocal polar curve is $n(n-1)-2\delta$.

121. Theorem:

The degree of the reciprocal of a curve with κ cusps is $n(n-1)-3\kappa$.

When the curve has a cusp, the first polar of any point not only passes through this cusp, but also has its tangent the same as the cuspidal tangent (§ 85). Therefore, the cusp counts as three among the intersections of the curve with its first polar. Consequently, the number of intersections is diminished by *three* for every cusp on the curve, and this diminution amounts to 3κ for the κ cusps on the curve. Hence, *the degree of the reciprocal curve is* $n(n-1)-3\kappa$.

Combining this with the preceding theorem, we obtain the following:—

The degree of the reciprocal polar of a curve with δ nodes and κ cusps is $n(n-1)-2\delta-3\kappa$.

122. Theorem:

If a curve has a multiple point of order k, the degree of its reciprocal polar curve is $n(n-1)-k(k-1)$.

We have seen that a multiple point of order k on a curve is a multiple point of order $k-1$ on the first polar (§ 86). Therefore, the multiple point counts as $k(k-1)$ among the intersections of the curve with its first polar. Consequently, the number of remaining intersections is $n(n-1)-k(k-1)$, *which is the degree of the reciprocal polar curve.**

* A multiple point of order k is equivalent to $\frac{1}{2}k(k-1)$ nodes, and for each node the class of a curve is diminished by two. Hence for a multiple point of order k, the class is diminished by $k(k-1)$, which agrees with the result we have established just now.

123. Envelopes:

If the equation of a curve involves a variable parameter, we obtain a series of different curves by giving different values to the parameter. All these curves touch a certain curve, which is called the *envelope* of the system.

Each curve is intersected by the consecutive curve in a set of points, and the locus of these points is called the envelope.

Let $f(x, y, z, \lambda)=0$ be the equation of a curve, which contains a single variable parameter λ. The parameter of a consecutive curve may be taken as $\lambda+\delta\lambda$, and it may be represented by the equation $f(x, y, z, \lambda+\delta\lambda)=0$, or $f_1(x, y, z, \lambda, \delta\lambda)=0$. Therefore, the equations $f=0$ and $f_1-f=0$ determine the co-ordinates of the points of intersection of two consecutive curves.

But $$f_1=f+\frac{\partial f}{\partial \lambda}\delta\lambda+\text{etc.}$$

$$\therefore \quad f_1-f=\frac{\partial f}{\partial \lambda}\delta\lambda+\text{etc.}$$

where $\delta\lambda$ is infinitesimal, and therefore its higher powers may be neglected. Therefore the equations

$$f=0, \qquad \text{and} \qquad \frac{\partial f}{\partial \lambda}=0$$

determine a set of points depending on the parameter λ.

Eliminating λ between these two equations, we obtain the equation of the locus of all points of intersection of consecutive curves of the system, which is the equation of the required envelope.

It may so happen that f is an integral and rational function of λ. In that case the above process is evidently equivalent to forming the discriminant of f regarded as a

function of λ, and then equating it to zero. Thus, if the quantic f, arranged in powers of λ, be written in the form—

$$a\lambda^n + nb\lambda^{n-1} + \frac{n(n-1)}{2!}c\lambda^{n-2} + \text{etc.} \qquad \dots \quad (1)$$

where a, b, c...are functions of (x, y, z), then the discriminant of (1) equated to zero will give the envelope.

If the curve $f(x, y, z, \lambda)=0$ passes through a fixed point (x', y', z'), we have $f(x', y', z', \lambda)=0$, and if this equation is solved for λ, there are found n values of λ, corresponding to each of which there is a curve of the system, and consequently, n curves of the system pass through any point. When, however, the point lies on the envelope, two of these curves coincide.

Ex. 1. The system of circles $(x-\lambda)^2+y^2=r,^2$ where λ is a parameter, has its centre on the axis of x. The envelope of this system is determined from $2(x-\lambda)=0$, *i.e.*, $x=\lambda$, which gives for the envelope $y^2=r^2$, or, $y=\pm r$, *i.e.*, two lines parallel to the locus of centres.

Ex. 2. The equation of the normal at any point $(a \cos\theta, b \sin\theta)$ on the ellipse

$$x^2/a^2+y^2/b^2=1$$

is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \qquad \dots \quad (1)$$

which involves a parameter θ.

Differentiating (1) with respect to θ, we obtain—

$ax \sec\theta . \tan\theta + by \operatorname{cosec}\theta . \cot\theta = 0$, or, $ax \sin^3\theta + by \cos^3\theta = 0 \dots$ (2)

Eliminating θ between (1) and (2) we obtain the equation of the envelope in the form—

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}},$$

which is called the *evolute* of the ellipse.

Ex. 3. To find the envelope of $at^n + bt^r + c = 0$

Differentiating with respect to t, we obtain $nat^{n-1} + pbt^{r-1} = 0$

whence $\qquad t^{n-r} = -\left(\frac{pb}{na}\right)$

Substituting this value of t in the equation we obtain the equation of the envelope in the form :

$$n^n a^r c^{n-r} \pm p^r (n-p)^{n-r} b^n = 0$$

where positive or negative sign is to be taken, according as n is odd or even.

Ex. 4. To find the envelope of the chords of curvature of the points on a conic.

The equation of the chord of curvature of the conic $x^2/a^2+y^2/b^2=1$ is

$$\frac{x}{a}\cos\alpha-\frac{y}{b}\sin\alpha=\cos 2\alpha$$

and therefore, the envelope becomes—

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-4\right)^3+27\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=0.$$

Ex. 5. Find the envelope of a circle of constant radius, whose centre moves on a given conic.

Let $x^2/a^2+y^2/b^2=1$ be the given conic, and $(x-\alpha)^2+(y-\beta)^2=r^2$ be the circle, the centre (α, β) moving on the conic.

Then we may write $\alpha=a\cos\theta,\quad \beta=b\sin\theta.$

The equation of the moving circle now becomes—

$$(a^2-b^2)\cos 2\theta-4ax\cos\theta-4by\sin\theta+2(x^2+y^2)+a^2+b^2-2r^2=0.$$

The envelope can now be found as already explained, and is called the curve *parallel* to the conic.

124. The Case of two Parameters:

It often happens that the equation of a curve contains two or more parameters connected by an equation or equations. In order to determine the envelope we make use of the method of indeterminate multiplier, the principles * of which we shall presently explain.

Let the equation $f(x, y, z, \lambda, \mu)=0$ of a curve contain two parameters λ, μ connected by a relation $\phi(\lambda, \mu)=0$.

Then we have

$$\frac{\partial f}{\partial \lambda}+\frac{\partial f}{\partial \mu}\cdot\frac{\partial \mu}{\partial \lambda}=0 \quad \text{and} \quad \frac{\partial \phi}{\partial \lambda}+\frac{\partial \phi}{\partial \mu}\cdot\frac{\partial \mu}{\partial \lambda}=0.$$

* The subject properly belongs to the province of Differential Geometry, and the student is referred to Edwards' Differential Calc., Chapter XI, for fuller treatment of the subject.

See also Cayley, Collected Works.

Eliminating $\frac{\partial \mu}{\partial \lambda}$ between these two equations, we obtain—

$$\psi \equiv \begin{vmatrix} \frac{\partial f}{\partial \lambda} & \frac{\partial f}{\partial \mu} \\ \frac{\partial \phi}{\partial \lambda} & \frac{\partial \phi}{\partial \mu} \end{vmatrix} = 0$$

The envelope is obtained by eliminating λ and μ between the equations $f=0$, $\phi=0$ and $\psi=0$.

We may eliminate one of the parameters between the given equations, and then obtain the envelope by the method of the preceding article.

A similar process is followed in the general case, when the k parameters in the equation of a curve are connected by $k-1$ relations. In this case the envelope is obtained by eliminating the k parameters and $k-1$ indeterminate multipliers between the $2k$ equations. The method may be best explained by means of a few simple illustrations.

Ex. 1. To find the envelope of lines which cut off intercepts on the axes, the sum of which is constant.

The equation of the line making intercepts a and b on the axes is

$$\frac{x}{a}+\frac{y}{b}=1$$

where $a+b=k$. (*Constant.*)

Differentiating the above equations, we have

$$-\frac{x}{a^2}-\frac{y}{b^2}\cdot\frac{\partial b}{\partial a}=0 \quad \text{and} \quad 1+\frac{\partial b}{\partial a}=0;$$

whence, $$-\frac{c}{a^2}=0, \text{ or, } \frac{\sqrt{x}}{a}=\pm\frac{\sqrt{y}}{b}$$

$$\therefore \quad a=\pm\sqrt{\frac{x}{y}}\cdot b$$

Substituting for a in the equation of the line, we obtain—

$$b = \pm x\frac{\sqrt{y}}{\sqrt{x}} + y = \pm\sqrt{xy} + y.$$

$$\therefore \pm\sqrt{\frac{x}{y}}(\pm\sqrt{xy} + y) + (\pm\sqrt{xy} + y) = k.$$

Or, $$(x \pm \sqrt{xy}) + (\pm\sqrt{xy} + y) = k.$$

i.e., $$(\sqrt{x} \pm \sqrt{y})^2 = k$$

$\therefore$ The envelope is $\sqrt{x} \pm \sqrt{y} = \pm\sqrt{k}$.

Ex. 2. Find the envelope of

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a and b are connected by the relation $ab = \text{const}$.

Differentiating the two equations, we obtain—

$$\frac{x}{a^2} + \frac{y}{b^2}\cdot\frac{\partial b}{\partial a} = 0, \quad \text{and} \quad b + a.\frac{\partial b}{\partial a} = 0.$$

$\therefore$ $\frac{x}{a^2} = \lambda b$ and $\frac{y}{b^2} = \lambda a$, where λ is an indeterminate multiplier.

Multiplying these by a and b respectively, and then adding, we obtain—

$$\frac{x}{a} + \frac{y}{b} = \lambda(ab + ab) = 2\lambda c \text{ (say)}$$

whence $1 = 2\lambda c$; and consequently $\lambda = \frac{1}{2c}$

$$\therefore \quad 2cx = a^2 b \quad \text{and} \quad 2cy = ab^2$$

and since $ab = c$, we have $2x = a$ and $2y = b$,

$$\therefore \quad 4xy = ab = \text{const}.$$

The envelope is $xy = \text{const.}$, *i.e.*, a hyperbola.

Ex. 3. The envelope of polars with respect to the circle $x^2 + y^2 = 2ax$ of points which lie on the circle $x^2 + y^2 = 2bx$ is

$$\{(a-b)x + ab\}^2 = b^2\{(x-a)^2 + y^2\}$$

125. We have seen that when the equation of a curve contains λ in the nth degree,

$$i.e., \text{ when } \quad f \equiv a\lambda^n + nb\lambda^{n-1} + \frac{n(n-1)}{2!} c\lambda^{n-2} + \ldots = 0$$

where $a, b, c, \ldots$ are functions of the co-ordinates, the envelope is obtained by equating to zero the discriminant* of f. Thus, for $n=2$, 3, etc., we have for the envelopes

$$ac - b^2 = 0$$

$$a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2 = 0, \text{ etc.}$$

It will be noticed that the degree of the envelope is $2(n-1)$ in the co-efficients $a, b, c\ldots$

Hence, when $a, b, c\ldots$ are linear, the degree of the envelope is $2(n-1)$.

The simplest of these cases is the envelope of

$$\text{A}\lambda^2 + 2\text{B}\lambda + \text{C} = 0$$

where A, B, C are all linear functions of the variables, so that the equation represents a right line.

Eliminating λ between

$$\text{A}\lambda^2 + 2\text{B}\lambda + \text{C} = 0 \quad \text{and} \quad \text{A}\lambda + \text{B} = 0$$

we obtain for the envelope $\text{B}^2 = \text{AC}$, which is a conic.

When A, B, C are expressions of the second degree, the envelope $\text{B}^2 = \text{AC}$ is a curve of the fourth degree, and so on.

* It has been proved by Prof. Cayley that the discriminant, in general, contains other loci besides the envelope. In fact the complete envelope of the variable curve consists of the proper envelope as explained above, together with the locus of the nodes of the variable curve *twice*, the locus of the cusps *thrice*, the envelope of the double tangents *twice* and the envelope of the stationary tangents *thrice*.

Cayley, Messenger of Mathematics, Vols. II and XII. See also Henrichi, Proc. of the London Math. Soc., Vol. II; J. M. Hill, *ibid*, Vol. XIX, and Salmon, H. P. Curves, § 89 (*a*).

Otherwise: The equation $A\lambda^2 + 2B\lambda + C = 0$ may be regarded as a quadratic equation to find the values of λ for the two particular members of the family, which pass through a given point (x, y). When (x, y) is a point on the envelope, these two members must coincide, and consequently, the quadratic in λ must have two equal roots, and the locus of such points is therefore $B^2 = AC$.

Ex. 1. The envelope of $x \cos \theta + y \sin \theta = \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}$ is

$$x^2/a^2 + y^2/b^2 = 1$$

For, the equation can be put into the form—

$$x + y \tan \theta = \sqrt{a^2 + b^2 \tan^2 \theta}$$

or, clearing the radical, $x^2 + y^2 \tan^2 \theta + 2xy \tan \theta = a^2 + b^2 \tan^2 \theta$

i.e., $$(y^2 - b^2) \tan^2 \theta + 2xy \tan \theta + (x^2 - a^2) = 0.$$

This, as a quadratic in $\tan \theta$, has two equal roots, if

$$4x^2y^2 = 4(x^2 - a^2)(y^2 - b^2), \quad i.e., \quad x^2/a^2 + y^2/b^2 = 1.$$

Ex. 2. Show that the envelope of the lines—

$$x \cos m\alpha = y \text{ is } m\alpha = a(\cos n\alpha)^{\frac{m}{n}}$$

where α is the arbitrary parameter, is the curve—

$$r^{\frac{n}{m-n}} = a^{\frac{n}{m-n}} \cos\left(\frac{n}{m-n}\right)\theta.$$

Ex. 3. Circles are described having for diameters the radii vectors from the origin to the curve $x^3 + y^3 = 3ax^2$. Prove that their envelope is the inverse of a semicubical parabola (Oxford, 1888).

Ex. 4. An equilateral triangle moves so that two of its sides pass through two fixed points P and Q. Prove that the envelope of the third side is a circle.

Ex. 5. Find the relation between a and b, when the envelope of the line $x/a + y/b = 1$ is the curve $x^r y^s = k^{r+s}$.

126. Evolutes:

The evolute of a curve may be defined as (*i*) the envelope of its normals, (*ii*) the locus of the intersection of consecutive normals, (*iii*) the locus of the centre of curvature at each point of the curve.

It is clearly seen that these definitions can be deduced one from the other.

In order to obtain the evolute of a curve, we generally take the equation of the normal at any point of the curve and then find its envelope. The same may be found by finding the co-ordinates of the centre of curvature at any point, and then determining the locus of the centre of curvature.

The methods will be best illustrated by means of the following simple examples:

Ex. 1. Find the evolute of the parabola $y^2 = 4ax$.

The normal at any point $(at^2, 2at)$ on the curve is—

$$tx + y = 2at + at^3 \qquad \dots \quad (1)$$

To obtain its envelope, we differentiate it with respect to t, and thus get

$$x = 3at^2 + 2a \qquad \dots \quad (2)$$

Eliminating t between (1) and (2), we obtain—

$$27ay^2 = 4(x-2a)^3$$

as the equation of the evolute of the parabola.

Ex. 2. Find the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

The normal at any point $(a\cos\theta, b\sin\theta)$ is—

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

The evolute of the ellipse, by Ex. 2, § 123, is, therefore

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

Ex. 3. Find the evolute of the Cissoid $(x^2+y^2)x=ay^2$

Writing the equation in the form $(a-x)y^2=x^3$, we may express the co-ordinates of any point on it in the form—

$$x=\frac{a}{(1+\theta^2)},\qquad y=\frac{a}{\theta(1+\theta^2)}$$

The equation of the normal is, therefore,

$$2\theta^3x+(1+3\theta^2)y=\frac{a(1+2\theta^2)}{\theta}$$

i.e.,
$$2\theta^4x+3\theta^3y-2\theta^2a+\theta y-a=0.$$

The discriminant of this is found to contain $(x+\frac{1}{2}a)^2+y^2$ as a factor, the remaining factor, therefere, gives the proper evolute, namely,

$$y^4+\frac{32}{3}a^2y^2+\frac{512}{27}a^3x=0.$$

Ex. 4. To find the evolute of the curve given by $x=ct$, $y=c/t$.

The centre of curvature at any point (t) is given by

$$x=c(3t^4+1)/2t^3,\qquad y=c(t^4+3)/2t.$$

Eliminating t between these, we obtain the equation of the evolute.

127. Normal of the Evolute:

The following construction for the normal of the evolute is useful:

Let I and J be any two finite points in the plane. Then, if the tangent at any point P of a curve meets IJ in M, and if M′ be the harmonic conjugate of M with respect to I, J, then the line PM′ may be regarded as the normal. From this it follows at once that if the point P be on the line IJ, then PM′ will coincide with that line. But when P coincides with either I or J, the points M, M′ coincide, and the normal coincides with the tangent.

Therefore, when I and J are circular points at infinity, and P is a point on IJ, the normal at P coincides with the line at infinity, and if the curve passes through either of the circular points at infinity, the normal coincides with the tangent.

128. When a curve is defined by its tangential equation, the line co-ordinates of its normal, and consequently the tangential equation of the evolute, can be easily obtained.

Let $\phi(\xi, \eta, \zeta)=0$ be the tangential equation of a curve. Then, if (ξ', η', ζ') be the line co-ordinates of any tangent,

$$\xi \frac{\partial\phi}{\partial\xi'} + \eta \frac{\partial\phi}{\partial\eta'} + \zeta \frac{\partial\phi}{\partial\zeta'} = 0 \qquad \dots \quad (1)$$

is the equation of its point of contact.

Let $\psi=0$ be the equation of a pair of points, I, J, then

$$\xi \frac{\partial\psi}{\partial\xi} + \eta \frac{\partial\psi}{\partial\eta'} + \zeta \frac{\partial\psi}{\partial\zeta'} = 0 \qquad \dots \quad (2)$$

is the equation of the pole of the given tangent with respect to IJ, *i.e.*, if P is the point where the tangent meets IJ, then the harmonic conjugate Q of P with respect to I, J is given by (2). If now I, J are the circular points at infinity, equation (2) gives the point at infinity on the normal.

Therefore the two equations (1) and (2) determine the line co-ordinates of the normal. If, therefore, ξ', η', ζ' be eliminated between the equations (1) and (2) and the equation of the curve, we obtain the tangential equation of the evolute.

The equation of the circular points at infinity is $\xi^2+\eta^2=0$.

Then $$\xi \frac{\partial\psi}{\partial\xi'} + \eta \frac{\partial\psi}{\partial\eta'} + \zeta \frac{\partial\psi}{\partial\zeta'} = 0$$

gives the condition of perpendicularity $\quad \xi\xi'+\eta\eta'=0$.

Ex. 1. Find the evolute of a central conic.

The tangential equation of such a conic is $\quad a^2\xi^2+b^2\eta^2=1$.

$\therefore$ The co-ordinates of the normal are determined by

$$a^2\xi\xi'+b^2\eta\eta'=1 \quad \text{and} \quad \xi\xi'+\eta\eta'=0$$

which give $$\xi\xi' = -\eta\eta' = \frac{1}{(a^2-b^2)}$$

Since (ξ', η') satisfy the equation of the conic, we must have—

$$a^2\xi'^2 + b^2\eta'^2 = 1.$$

Eliminating ξ', η' between these equations, the required evolute is

$$\frac{a^2}{\xi^2} + \frac{b^2}{\eta^2} = (a^2-b^2)^2.$$

Ex. 2. Find the tangential equation of the evolute of the parabola

$$a\eta^2 = \xi\zeta.$$

Ex. 3. Obtain the evolute of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Any point on this curve is $x = a\cos^3\theta$, $y = a\sin^3\theta$.

The normal at that point is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

The envelope of this will be $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$, which is the required evolute.

129. Caustics:

There is a class of curves, called Caustics, the investigation of which, although originally suggested by the science of optics, belongs purely to the theory of curves. The subject formed one of the earliest questions discussed in connection with envelopes,* and as such we shall discuss here some of the simplest and interesting cases.

Definition:

Let P be a point in the plane of a curve, which may be considered as the boundary separating two optical media. If a ray of light from P be incident on the curve, the reflected or refracted ray envelopes a curve which, in general, is called a *Caustic*—*Katacaustic* in case of reflection and *Diacaustic* in case of refraction.

* The subject was introduced by Tschirnhausen, *Acta Eruditorum* (1682), referred to by Gregory, Examples, p. 224.

Gergonne * has proved that each katacaustic is the evolute of an algebraic curve which, according to Quetelet † is called the *Secondary Caustic* or *Anticaustic*.

Quetelet has given a practical method by which caustics may be regarded as evolutes, and in fact, he gives the following construction :

If with each point successively of the reflecting curve as centre, and its distance from the radiant point as radius, we describe a series of circles, the envelope of all these circles will be a curve, the evolute of which is the katacaustic required.

In a like manner, Quetelet has given the following theorem :

If with each point successively of the refracting curve as centre, and a length in a constant ratio to its distance from the radiant point as radius, we describe a series of circles, the envelope of all these circles will be a curve whose evolute is the Diacaustic.

130. Equation of Katacaustics :

Let $F(x, y)=0$ be the equation of the reflecting curve, and $P(x', y')$ be a luminous point in its plane.

Let $Q(\alpha, \beta)$ be the point of incidence of a ray on the curve F. If then $T=0$ and $N=0$ be the equations of the tangent and normal to the curve at Q, the equation of the incident ray PQ may be written as $T+\lambda N=0$, which is satisfied by (x', y'), *i.e.*, PQ is the line $TN'-T'N=0$, where T' and N' are the results of substituting x', y' for x and y respectively in T and N.

Now, the incident ray and the reflected ray are equally inclined to the normal. Therefore, the reflected ray is the

* Gergonne, Ann. de Math., t. 15 (1825), p. 345.

† Quetelet, Brux. Ac. Nouv. Mém., Vol. 5 (1829), Nr. 1.

fourth harmonic to the three lines, *viz.*, T, N and TN′ − T′N. Hence the equation of the reflected ray is—

$$TN' + T'N = 0$$

which involves (α, β) as parameters connected by the relation $F(\alpha, \beta) = 0$.

Thus, by the usual method we can now determine the envelope of this reflected ray which is the *Katacaustic*.

Ex. Caustic by reflection of a straight line :

If the reflecting line be taken as the x-axis, and the radiant point P be taken on the y-axis at a distance a from the origin O, then it is evident that the reflected rays all pass through a fixed point Q on the y-axis on the other side at a distance a from O. Q is said to be the *reflection* of P about the line.

131. Caustic by Reflection of a Circle :

The actual determination of caustics by reflection or katacaustics of general curves presents some difficulties, but they can be very easily calculated for simpler curves, for instance, a straight line or a circle. We shall here determine the caustic of a circle.*

Let $x^2 + y^2 = r^2$ be the equation of the reflecting circle, (α, β) be the co-ordinates of the radiant point P, (a, b) those of the point of incidence Q, and (x, y) the co-ordinates of any point on the reflected ray.

Since $Q(a, b)$ is a point on the circle, we may take

$$a = r \cos\theta, \quad b = r \sin\theta.$$

The tangent at Q is then

$$x \cos\theta + y \sin\theta = r \qquad \ldots \quad (1)$$

* A very elegant solution of the problem is given by Lagrange in the Mém de Turin. Mr. P. Smith discussed the same in a note in the Cambridge and Dublin Mathematical Journal, Vol. 2 (1847), p. 237. Prof. Cayley investigated the problem very exhaustively in his well-known paper—" A Memoir upon Caustics "—Coll. Works, Vol. 2, pp. 336-380.

and the normal is the line

$$x \sin \theta - y \cos \theta = 0 \qquad \ldots \quad (2)$$

The equation of the reflected ray is, therefore, by the preceding article,

$$(\alpha \cos \theta + \beta \sin \theta - r)(x \sin \theta - y \cos \theta)$$
$$+ (\alpha \sin \theta - \beta \cos \theta)(x \cos \theta + y \sin \theta - r) = 0$$

which may be written in the form—

$$(\alpha y + \beta x) \cos 2\theta + (\beta y - \alpha x) \sin 2\theta$$
$$+ r(x + \alpha) \sin \theta - r(y + \beta) \cos \theta = 0 \quad \ldots \quad (3)$$

where θ is a variable parameter.

The envelope of this, by the usual method, is found to be

$$[4(\alpha^2 + \beta^2)(x^2 + y^2) - r^2\{(x + \alpha)^2 + (y + \beta)^2\}]^3$$
$$= 27(\beta x - \alpha y)^2(x^2 + y^2 - \alpha^2 - \beta^2)^2 \quad \ldots \quad (4)$$

which is the equation of the caustic by reflection of a circle and was first obtained by St. Laurent.

If, however, the axis of x passes through the radiant point, and the radius of the circle be taken equal to unity, we have $\beta = 0$ and $r = 1$, and the equation of the caustic reduces to

$$\{(4\alpha^2 - 1)(x^2 + y^2) - 2\alpha x - \alpha^2\}^3$$
$$= 27\alpha^2 y^2 (x^2 + y^2 - \alpha^2)^2 \qquad \ldots \quad (5)$$

The equation (3) of the reflected ray reduces to the form

$$(-2\alpha \cos \theta + 1)\, x + \frac{\alpha \cos 2\theta - \cos \theta}{\sin \theta}\, y + \alpha = 0 \qquad \ldots \quad (6)$$

Differentiating this with respect to θ, we obtain—

$$(-2\alpha \sin \theta) x + \frac{-\alpha \cos \theta (1 + 2 \sin^2 \theta) + 1}{\sin^2 \theta}\, y = 0$$

whence, $$x=\frac{a^2\cos\theta(1+2\sin^2\theta)-a}{1-3a\cos 2\theta+2a^2}$$

$$y=\frac{2a^2\sin^3\theta}{1-3a\cos 2\theta+2a^2}$$

i.e., the co-ordinates (x, y) of any point on the caustic are expressed in terms of the angle θ, which is the parameter of the point of incidence.

132. Tangential Equation of the Caustic:

If the equation (6) of the reflected ray be put into the form $\xi x+\eta y+a=0$, then we must have—

$$\xi=-2a\cos\theta+1$$

$$\eta=\frac{a\cos 2\theta-\cos\theta}{\sin\theta}$$

whence, $$(\xi-1)^2-4a^2=-4a^2\sin^2\theta$$

$$\xi^2+\eta^2=\frac{1}{\sin^2\theta}(1-2a\cos\theta+a^2)$$

and $$\xi+a^2=1-2a\cos\theta+a^2.$$

Therefore,

$$(\xi^2+\eta^2)\{(\xi-1)^2-4a^2\}+4a^2\xi+4a^4=0$$

i.e., $$\{\xi(\xi-1)-2a^2\}^2+\eta^2\{(\xi-1)^2-4a^2\}=0 \quad \ldots \quad (7)$$

which may be considered as the tangential equation of the caustic by reflection of a circle.

If, however, (ξ, η) be considered as the co-ordinates of a point, then the equation (7) may be regarded as the equation of the reciprocal polar of the caustic (§ 114).

133. If we put $y=0$ in the equation (5) of the caustic, we obtain

$$\{(4a^2-1)x^2-2ax-a^2\}^3=0$$

i.e., $$x = \frac{-a}{2a+1}, \quad x = \frac{a}{2a-1}$$

or, the caustic meets the axis of x in two points each of which is a triple point.

If, again, we put $x^2+y^2=a^2$, we obtain—

$$\{(4a^2-1)a^2-2ax-a^2\}^3=0$$

which gives $$x=-a(1-2a^2)$$

and, therefore, $$y=\pm 2a^2\sqrt{1-a^2}$$

or, the caustic meets the circle $x^2+y^2-a^2=0$ in two points, each of which is a triple point.

The nature of the infinite branches can be found by considering the highest degree terms, *i.e.*, the terms of degree six and five. The two asymptotes are thus found to be given by—

$$y=\frac{(4a^2-1)^{\frac{3}{2}}}{\sqrt{1-a^2}\ (8a^2+1)}\left\{x-\frac{3a}{4a^2-1}\right\}.$$

By considering the equation of the reflected ray (6), the tangents parallel and perpendicular to the axis of x can be found easily.

Thus, the parallel tangents are—

$$y=\pm\frac{\sqrt{4a^2-1}}{2a}$$

and the tangents perpendicular to the axis of x are—

$$x=\frac{-2a}{1\mp\sqrt{8a^2+1}}$$

which are, in fact, bitangents of the caustic.

Again, when the radiant point lies on the circle, $a=1$, and the equation of the curve reduces to—

$$\{3y^2+(x-1)(3x+1)\}^3=27y^2(x^2+y^2-1)^2$$

which is divisible by $(x-1)^2$. Removing the factor, the caustic is a curve of the fourth order, *viz.*,

$$27y^4+18y^2(3x^2-1)+(x-1)(3x+1)^3=0.$$

In the case of parallel rays, the radiant point is at infinity, *i.e.*, $a=\infty$, and the caustic reduces to—

$$(4x^2+4y^2-1)^3-27y^2=0$$

which is a sextic.

We thus see that the caustic by reflection of a circle is a curve of order 6, has 4 nodes, 6 cusps (including the circular points), etc. For detailed investigation, see Prof. Cayley's paper on Caustics, Coll. Works, Vol. II, p. 357. Also a Memoir by Rev. Hamnet Holditch, Quarterly Math. Journal, Vol. I (1857), pp. 93-111.

Ex. 1. If the incident rays are parallel, show that the caustic is an *epicycloid* formed by the rolling of one circle upon another of twice its radius.

Ex. 2. When the incident rays diverge from a point on the circumference of the reflecting circle, show that the caustic curve is a *cardioid*, which is formed by the rolling of a circle upon another of equal radius.

Ex. 3. Rays diverging from the focus of a parabola are reflected from its evolute. Prove that the secondary caustic is a parabola.

Ex. 4. Rays parallel to the axis of y are reflected from the curve $y=e^x$, show that the caustic is the curve

$$y=\tfrac{1}{2}\{e^{x+1}+e^{-(x+1)}\}.$$

Ex. 5. Obtain the caustic by reflection of an ellipse, the radiant point being at the centre.

134. Intersection of the Caustic with the Reflecting Circle:

If in the equation of the caustic, we put $x^2+y^2=1$, we have—

$$(3a^2-1-2ax)^3-17a^2y^2(1-a^2)^2=0$$

i.e., $(3a^2-1-2ax)^3-27a^2(1-x^2)(1-a^2)^2=0$

which reduces to the form—

$$(ax-1)^2(8ax-27a^4+18a^2+1)=0.$$

The factor $(ax-1)^2$ equated to zero shews that the caustic touches the circle at the points

$$x=\frac{1}{a}, \quad y=\pm\sqrt{1-\frac{1}{a^2}}$$

i.e., at the points where the circle is met by the polar of the radiant point. The other factor gives—

$$x=\frac{27a^4-18a^2-1}{8a}.$$

It can be very easily shewn that the curve passes through the circular points at infinity, which are cusps on the curve and the points where the axis of x meets the curve are cusps (the axis of x being the tangent) and the two points of intersection with the circle $x^2+y^2-a^2=0$ are also cusps, the tangent at each point coinciding with the tangent of the circle. There are thus in all 6 cusps.

135. Caustic by Refraction of a Straight Line:

Let the line be taken as the axis of y, and let the axis of x pass through the radiant point P, so that P is the point $(-1, 0)$.

Let ϕ and ϕ' be the angles of incidence and refraction respectively.

Then, $\sin\phi : \sin\phi' = \mu$ (refractive index)

$$=\frac{1}{k} \text{ (say)}.$$

The equation of the incident ray is—

$$y=\tan\phi(x+1).$$

Let the refracted ray be

$$y = x \tan \phi' + c.$$

Since both the rays meet on the line $x=0$, we must have—

$$c = \tan \phi$$

and the refracted ray becomes—

$$y - x \tan \phi' = \tan \phi$$

or,
$$y - \frac{k \sin \phi}{\sqrt{1-k^2 \sin^2 \phi}} x - \tan \phi = 0 \qquad \dots \quad (1)$$

Differentiating this with respect to ϕ, and combining the two equations, we obtain, after a simple reduction—

$$kx = -\frac{(1-k^2\sin^2\phi)^{\frac{3}{2}}}{\cos^3\phi}$$

$$k'y = -\frac{k'^3 \sin^3 \phi}{\cos^3 \phi}$$

where $\quad k' = \sqrt{1-k^2}$

Hence, eliminating ϕ, we obtain—

$$(kx)^{\frac{2}{3}} - (k'y)^{\frac{2}{3}} = 1$$

which is the required equation of the caustic.

When $k<1$, *i.e.*, refraction takes place into a denser medium, k'^2 is positive, the caustic is the evolute of a hyperbola.

But when $k>1$, *i.e.*, refraction takes place in a rarer medium, k'^2 is negative and the caustic is the evolute of an ellipse (§ 126, Ex. 2). Thus the hyperbola or the ellipse is the Anticaustic or the Secondary Caustic.

From what has been said in § 129 we can easily form the equation of the secondary caustic as follows :—

The equation of the variable circle may be taken as

$$x^2+(y-\tan\phi)^2-k^2\sec^2\phi=0$$

the envelope of which is found to be—

$$k'^2(x^2+y^2-k^2)-y^2=0$$

i.e.,
$$k'^2x^2-k^2y^2-k^2k'^2=0$$

or,
$$\frac{x^2}{k^2}-\frac{y^2}{k'^2}=1$$

which is the equation of the Secondary Caustic. It will be seen that the radiant point is a focus of the conic.

136. Secondary Caustic :

We have seen that the caustic by reflection or refraction may be regarded as the evolute of a certain envelope which is called the secondary caustic. In fact, the reflected or refracted rays are the normals to a series of secondary caustics; any one of these has the reflected or refracted rays for normals, and consequently the caustic curve for evolute.

It is usually more convenient to find a secondary caustic in some cases than the caustic itself; for instance, in the preceding article, the secondary caustic by refraction of a straight line is an ellipse or a hyperbola.

We shall now determine the secondary caustic by refraction at a circle.

Let $x^2+y^2=r^2$ be the refracting circle, $P(\alpha, \beta)$ the radiant point, and (ξ, η) any point on the circle, so that

$$\xi=r\cos\theta, \quad \eta=r\sin\theta.$$

If μ be the refractive index, the secondary caustic will be the envelope of the variable circle—

$$\mu^2\{(x-r\cos\theta)^2+(y-r\sin\theta)^2\}$$
$$-\{(\alpha-r\cos\theta)^2+(\beta-r\sin\theta)^2\}=0.$$

Writing the equation in the form—

$$\mu^2(x^2+y^2+r^2)-(\alpha^2+\beta^2+r^2)$$
$$-2r(\mu^2x-\alpha)\cos\theta-2r(\mu^2y-\beta)\sin\theta=0,$$

the envelope becomes—

$$\{\mu^2(x^2+y^2+r^2)-(\alpha^2+\beta^2+r^2)\}^2$$
$$=4r^2\{(\mu^2x-\alpha)^2+(\mu^2y-\beta)^2\}$$

or,
$$\{\mu^2(x^2+y^2-r^2)-(\alpha^2+\beta^2-r^2)\}^2$$
$$=4r^2\mu^2\{(x-\alpha)^2+(y-\beta)^2\}.$$

If now the axis of x is taken to pass through the radiant point, $\beta=0$ and $\alpha=a$ (say), then the equation becomes—

$$\{\mu^2(x^2+y^2-r^2)-a^2+r^2\}^2=4r^2\mu^2\{(x-\alpha)^2+y^2\}$$

which, after simplification, may be put into the form—

$$\mu\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}=\sqrt{(x-a)^2+y^2}+r\left(\mu-\frac{1}{\mu}\right).$$

The above equation of the secondary caustic is evidently of the form—

$$m\rho+m'\rho'=c\,(\text{say}),$$

where ρ and ρ' are the distances of the point (x, y) on the locus from the two fixed points

$$\left(\frac{a}{\mu^2}, 0\right) \text{ and } (a, 0)$$

respectively, and $\quad \mu=-m:m'$.

Thus, the locus of (x, y), *i.e.*, the secondary caustic is the Oval of Descartes or the Cartesian,* of which the two fixed points are the foci. Therefore *the Caustic by refraction of a circle is the evolute of a Cartesian Oval.*

* See Williamson's Diff. Calculus, Chap. XX, pp. 375-82.

The cases of parallel rays or of rays proceeding from a point on the circumference of the circle may be deduced from the above results by assuming $a=\infty$ or $a=r$ respectively. For further discussions the student is referred to *Geometrical Optics* by R. S. Heath, Chapter VI, pp. 99-133, and to Prof. Cayley's Memoir.

Ex. 1. Prove that the form of the caustic curve near the cusp is a semi-cubical parabola.

Ex. 2. Find the caustic by refraction of a circle, when the incident rays are parallel.

Ex. 3. Rays diverging from the centre of a given circle are refracted at a curve so that the refracted rays are all tangents to the circle. Find the equation to the refracting curve.

Ex. 4. Show that the caustic by refraction of a circle when the radiant point is on the circumference is also the caustic by reflection for the same radiant point and for a reflecting circle concentric with the refracting circle.

Ex. 5. Prove that the caustic by reflexion of a circle is the evolute of the *limaçon*.

137. Pedal Curves:

The locus of the foot of the perpendicular drawn from any origin O on to the tangent at any point of a curve is called the *first positive pedal* of the curve with respect to the origin.

The pedal of the first positive pedal is called the *second positive pedal*, the pedal of this latter is called the *third positive pedal*, and so on.

The curve which has the original curve for its first positive pedal is called the first *negative* pedal, and so on.

Let OY be the perpendicular on the tangent at any point P of a curve, and OZ be the perpendicular drawn from O on to the tangent at Y to the locus of Y; then the angle OPY = angle OYZ, or, in other words:—

The angles between the radius vector and the tangent at corresponding points of a curve and its pedal are equal.

In the same case, we have $OP.OZ = OY^2$. Let OY and OY$'$ be the perpendiculars drawn from O on to the tangents at two consecutive points P and P$'$ of the curve, the tangents meeting at the point T.

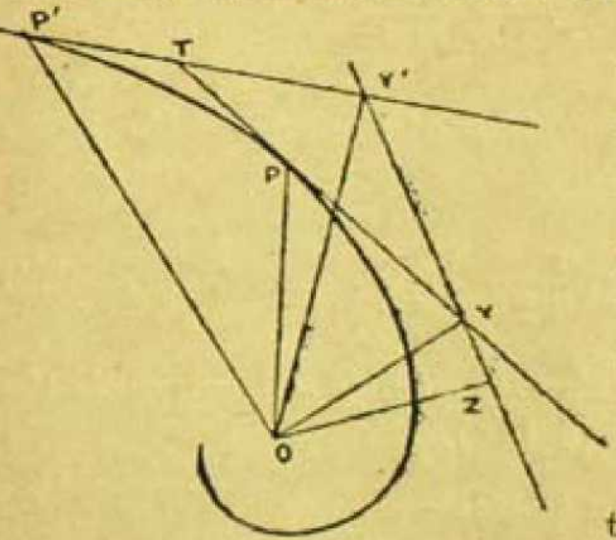
It is clear then, since the angle $YOY' = YTY'$, the points O, Y, Y$'$, T are concyclic, and therefore,

$$OYZ = \pi - OYY' = OTY'.$$

Hence, the triangles are similar, and we have—

$$\frac{OZ}{OY} = \frac{OY'}{OT} \qquad \dots \quad (1)$$

In the limit when P and P$'$ coincide, the angle OTY$'$ becomes equal to the angle OPY, and $OY' = OY$, $OT = OP$.



Hence, $OPY = OYZ$ and $OP.OZ = OY^2$, and the theorem is proved.

In the limit when P and P$'$ coincide, the circle through OYY$'$T has OP as diameter and touches the tangent YY$'$ to the pedal.

Hence we obtain the theorem that *the circle on radius vector as diameter touches the pedal.*

138. The Cartesian Equation of the Pedal:

Let $\phi(\xi, \eta) = 0$ be the tangential equation of a curve. Let any tangent to this curve cut the axes of x and y in P and Q respectively.

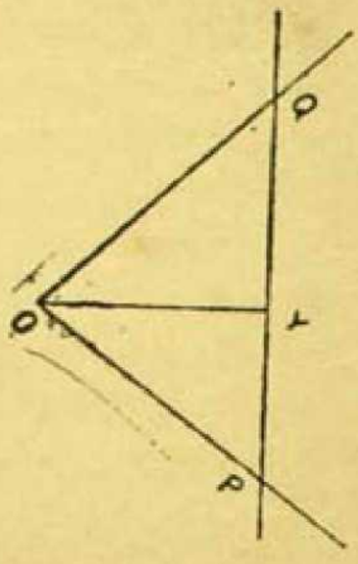
Let (x, y) be the co-ordinates of Y, the foot of the perpendicular from O on to AB.

If $\angle POY = \alpha$, we have—

$OY = OP \cos\alpha = OQ \sin\alpha.$

But $OP = 1/\xi$, $OQ = 1/\eta$;

$$\therefore \frac{\cos\alpha}{\xi} = \frac{\sin\alpha}{\eta} = \frac{1}{\sqrt{\xi^2 + \eta^2}}$$



Then $x = OY \cos\alpha = OP \cos^2\alpha$

$y = OY \sin\alpha = OQ \sin^2\alpha$

i.e., $$x = \frac{\xi}{\xi^2 + \eta^2} \quad \text{and} \quad y = \frac{\eta}{\xi^2 + \eta^2}$$

Hence, $$\xi = \frac{x}{x^2 + y^2}, \quad \eta = \frac{y}{x^2 = y^2}$$

Substituting these in the equation of the curve, the locus of Y becomes—

$$\phi\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) = 0 \qquad \ldots \quad (1)$$

The inverse of (1) is evidently $\phi(x, y) = 0$, which is the polar reciprocal of the curve (§ 114).

Hence, we have the definition :—

The polar reciprocal of a curve is the inverse of the first positive pedal, and the pedal is the inverse of the polar reciprocal curve.

Cor. The tangential equation of the first negative pedal is—

$$\phi\{\xi/(\xi^2 + \eta^2), \ \eta/(\xi^2 + \eta^2)\} = 0$$

when $\phi(x, y) = 0$ is the point-equation of the curve.

Ex. 1. The first positive pedal of the curve $(x/a)^n+(y/b)^n=1$ is

$$(x^2+y^2)^{\frac{n}{n-1}}=(ax)^{\frac{n}{n-1}}+(by)^{\frac{n}{n-1}}.$$

Ex. 2. Show that the first positive pedal of $x^3+y^3=a^3$ is

$$(x^2+y^2)^{\frac{3}{2}}=a^{\frac{3}{2}}(x^{\frac{3}{2}}+y^{\frac{3}{2}}).$$

Ex. 3. If the origin be situated on a curve, prove that its first pedal has a cusp at the origin.

Ex. 4. The pedal of the parabola $y^2=4ax$ with respect to the vertex is the cuspidal cubic (the Cissoid)

$$x(x^2+y^2)+ay^2=0.$$

Ex. 5. The pedal of a conic *w.r.t.* its centre is a unicursal quartic.

Ex. 6. The first positive pedal of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is

$$r=\pm a\sin\theta\cos\theta.$$

Ex. 7. The negative pedal of an ellipse *w. r. t.* a focus as pole is a quartic having the circular lines as stationary tangents. (Salmon, H. P. Curves, Ex. 3, p. 107.)

139. Inverse Curves :

In § 15, we have discussed the process of circular inversion, by means of which, from a given curve, another curve, called its *inverse*, can be derived. The principles have been explained by means of simple illustrations. We shall now discuss some of the properties for a general curve of order n.

Let $\qquad f\equiv u_0+u_1+u_2+\ldots+u_n=0 \qquad \ldots \quad (1)$

be the equation of the given *n-ic*.

Then, by the formulae of § 15, the equation of the inverse becomes—

$$u_0(x^2+y^2)^n+k^2u_1(x^2+y^2)^{n-1}+\ldots+k^{2n}u_n=0 \quad \ldots \quad (2)$$

which shows that the origin is a multiple point of order n on the inverse curve. The tangents at the multiple point

are evidently given by $u_n = 0$, which shows that the tangents $u_n = 0$ are parallel to the asymptotes of the original curve.

The equation (2) shows that the inverse curve has also a multiple point of order n at each of the circular points at infinity.

The following examples will illustrate some of the properties of inverse curves, and can easily be solved with the help of the formulæ established in § 15.

Ex. 1. Find the inverse of a conic *w. r. t.* a focus as the pole.

Ex. 2. Prove that the inverse of the curve $\alpha\rho_1 + \beta\rho_2 + \gamma\rho_3 = 0$ *w.r.t.* any origin is a curve whose equation is of the same form.

Ex. 3. Show that to a double point on any curve corresponds another double point of the same kind on the inverse curve with respect to any origin.

Ex 4. Show that the inverse to the k-th positive pedal is the k-th negative pedal of the inverse curve.

Ex. 5. The osculating circle at any point of a curve inverts, in general, into the osculating circle of the inverse curve at the inverse point.

Ex. 6. Discuss the case when the osculating circle passes through the origin.

Ex. 7. Find the number of osculating circles of a given curve which passes through a given point.

140. Parallel Curves:

Definition. The envelope of a line parallel to the tangent of a given curve at a fixed distance is called a curve parallel to the given one.

From this definition it follows that, if on the normal at any point P to a given curve, a point Q is taken such that

$$PQ = k = \text{const.}$$

the locus of Q is a curve parallel to the given curve. It follows then that all parallel curves have the same

normals and the same evolute; but every normal to a parallel curve is normal in two places corresponding to the values $\pm k$.

The two possible positions of Q corresponding to the values $\pm k$ are, in general, branches of the same curve. But they may be different curves in certain cases.

Let $$\xi x + \eta y + \zeta z = 0$$

be the equation of a tangent to a curve. Then, the equation of a parallel line at a distance k from it may be written as

$$\xi x + \eta y + \zeta + k\sqrt{\xi^2 + \eta^2} = 0$$

the envelope of which will be a parallel curve. If then $\phi(\xi, \eta, \zeta) = 0$ be the tangential equation of a curve, that of the parallel curve is obtained by writing

$$\zeta + k\sqrt{\xi^2 + \eta^2}$$

for ζ in the given equation, *i.e.*, the equation of the parallel curve is—

$$\phi(\xi, \eta, \zeta + k\sqrt{\xi^2 + \eta^2}) = 0$$

A parallel curve may also be regarded as the envelope of a circle of given radius whose centre moves along the curve. The methods will be clearly illustrated by means of a few examples.

Ex. 1. Find the parallel to the curve—

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

The co-ordinates of any point on this curve may be taken as—

$$a \cos^3\theta, \ a \sin^3\theta$$

The equation of the tangent is then—

$$x \cos\theta + y \sin\theta = a \sin\theta \cos\theta$$

and that of a line parallel to this at a distance k, is—

$$x \cos\theta + y \sin\theta = k + a \sin\theta \cos\theta.$$

The envelope of this line is found to be—

$$\{3(x^2+y^2-a^2)-4k^2\}^3+\{27axy-9k(x^2+y^2)-18a^2k+8k^3\}^2=0.$$

In this case there are two different curves constituting the parallel and these are not branches of the same curve.

The tangential equation is found to be—

$$(\xi^2+\eta^2)\zeta^2=\{a\xi\eta\pm k(\xi^2+\eta^2)\}$$

Ex. 2. In the case of a circle of radius r, the parallel becomes two concentric circles, of radii $r\pm k$, and these are certainly different curves.

Ex. 3. To find the tangential equation of the parallel to

$$x^2/a^2+y^2/b^2=1.$$

The tangential equation of the ellipse is $a^2\xi^2+b^2\eta^2=\zeta^2$, and consequently, that of the parallel is—

$$a^2\xi^2+b^2\eta^2=(\zeta+k\sqrt{\xi^2+\eta^2})^2$$

or, $$\{(a^2-k^2)\xi^2+(b^2-k^5)\eta^2-\zeta^2\}^2=4k^2(\xi^2+\eta^2)\zeta^2.$$

Ex. 4. Show that the tangential equation of the parallel to the parabola $y^2=4ax$ is

$$(a\eta^2-\xi\zeta)^2=k^2\xi^2(\xi^2+\eta^2).$$

Ex. 5. The radii of a co-axial system of circles are increased by a constant. Show that their new envelope is parallel to their old.

141. Isoptic Loci :

The locus of the point of intersection of two tangents which cut one another at a given angle is called an *Isoptic Locus* of the given curve. If the tangents cutting at an angle $\pi-\alpha$ are included among those cutting at the angle α, the locus is an algebraic curve.

The method of finding the locus will be best illustrated by the following examples :

Ex. 1. Find the isoptic locus of the parabola $y^2=4ax$.

Let α be the angle at which the tangents cut one another.

For all values of m, the line $y=mx+\dfrac{a}{m}$ is a tangent to the parabola.

Hence the slopes of the two tangents passing through any point (x', y') are given by—

$$m^2x' - my' + a = 0.$$

If m_1 and m_2 are the slopes, then—

$$m_1 + m_2 = \frac{y'}{x'} \quad \text{and} \quad m_1m_2 = \frac{a}{x'}$$

$$\therefore \tan\alpha = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{\sqrt{y'^2/x'^2 - 4a/x'}}{1 + \dfrac{a}{x'}} = \frac{\sqrt{y'^2 - 4ax'}}{a + x'}$$

$\therefore$ The locus of (x', y') becomes—

$$y^2 - 4ax = \tan^2\alpha(a + x)^2.$$

Ex. 2. Find the isoptic locus for $x^2/a^2 + y^2/b^2 = 1$.

The tangential equation of the ellipse is—

$$a^2\xi^2 + b^2\eta^2 = \zeta^2 \qquad \dots \quad (1)$$

If we eliminate ζ between this and $\xi x + \eta y = \zeta$, the resulting equation is

$$(a^2 - x^2)\xi^2 + (b^2 - y^2)\eta^2 - 2xy\xi\eta = 0 \qquad \dots \quad (2)$$

If now $m = \tan\theta = -\xi/\eta$ be the slope of the tangent drawn from any point (x, y), we obtain from (2) the equation—

$$m^2(a^2 - x^2) + 2xym + (b^2 - y^2) = 0,$$

which gives the slopes of the two tangents.

Now, m_1 and m_2 being the slopes, we obtain—

$$m_1 + m_2 = \frac{-2xy}{a^2 - x^2}, \quad m_1m_2 = \frac{b^2 - y^2}{a^2 - x^2}$$

whence $$-\tan\alpha = \frac{\sqrt{4b^2x^2 + 4a^2y^2 - 4a^2b^2}}{x^2 + y^2 - a^2 - b^2}$$

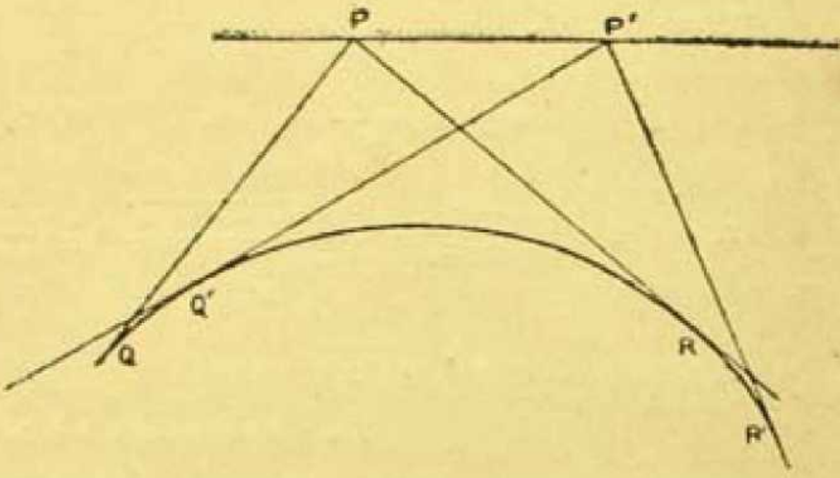
i.e., the equation of the locus is—

$$4(b^2x^2 + a^2y^2 - a^2b^2) = (x^2 + y^2 - a^2 - b^2)^2 \tan^2\alpha.$$

The case of a general n-ic assumes a simpler form, when the constant angle is a right angle, and will be taken up shortly.

142. A Theorem :

If PQ *and* PR *are two tangents to a curve nclined at a constant angle, the circle* PQR *touches the isoptic locus .*



Let P and P′ be consecutive points on the isoptic locus, and PQ, PR and P′Q′ P′R′ be tangents inclined at an angle α to each other. Then, Q, P, P′, R are concyclic, since $\angle QPR = \angle QP'R$, and in the limit, the circle PQR touches the locus of P.

It is clear then that if the tangents PQ and PR to a curve are inclined at a constant angle and the normals at Q and R meet at S, then PS is the normal to the isoptic locus.

Since QS and RS are perpendicular to the chords PQ, and PR of the above circle, the point S lies on this circle, and PS is a diameter and is consequently perpendicular to the tangent at P to the circle and the isoptic locus, that is to say, PS is the normal at P to the isoptic locus.

It can be easily shown that the isoptic locus has $m(m-1)$-ple points at I and J.

143. Orthoptic Loci :*

The locus of the point of intersection of two tangents to a curve which cut one another at right angles is called the *orthoptic locus* of the curve.

This is a particular class of isoptic locus, when the constant angle is a right angle.

* Dr. C. Taylor—" Note of a Theory of Orthoptic and Isoptic Locus," Proc. of the Royal Soc., London, Vol. 37 (1884), pp. 138-141.

Cartesian Equation :

Let $\phi(\xi, \eta)=0$ be the tangential equation of the curve, and $\xi x+\eta y+1=0$, that of a point P on the locus, so that two of the tangents drawn from P to the curve are perpendicular. If now we make $\phi(\xi, \eta)=0$ homogeneous in ξ, η by means of $\xi x+\eta y+1=0$, the resulting equation in ξ/η gives the slope of the tangents which can be drawn from P to the curve.

Putting $-\xi/\eta=m$, the resulting equation can be put into the form
$$\psi(m)=0 \qquad \dots (1)$$

If two tangents are at right angles, two of the roots m_1, m_2 of (1) must be connected together by the relation
$$m_1 m_2+1=0$$

The condition for this is that the eliminant of
$$\psi(m)=0 \quad \text{and} \quad \psi(-m^{-1})=0$$
should vanish, which gives a relation between x and y, and this is the orthoptic locus required.

Since this eliminant for a curve of the mth class is of degree $(m-1)$ in the co-efficients, which are themselves, in general, of degree m, the degree of the orthoptic locus of a curve of the mth class cannot be greater than $m(m-1)$.

Ex. 1. Find the orthoptic locus of the parabola $y^2=4ax$.

This follows from *Ex.* 1, § 141, when
$$a=\frac{\pi}{2}.$$

By using the method of the present article, we proceed as follows :—

The tangential equation of the parabola is $a\eta^2=\xi$.

Making this homogeneous, we get
$$a\eta^2+\xi(\xi x+\eta y)=a\eta^2+x\xi^2+y\xi\eta=0.$$

Putting $-\xi/\eta=m$, we obtain $m^2x-my+a=$ $\dots$ (1)

Putting $m=-m^{-1}$, we obtain—
$$m^2a+my+x=0 \qquad \dots (2)$$

From (1) and (2), we have—

$$\frac{m^2}{-xy-ay}=\frac{m}{a^2-x^2}=\frac{1}{xy+ay}$$

whence $$\{(a-x)^2+y^2\}(x+a)^2=0$$

which gives $x+a=0$ as the locus required.

Ex. 2. The orthoptic locus of

$$x^2/a^2+y^2/b^2=1$$

is found to be $x^2+y^2=a^2+b^2$, which is the director circle.

Ex. 3. In the case of the circle $x^2+y^2=r^2$, it becomes $x^2+y^2=2r^2$ which is a concentric circle.

Ex. 4. Find the orthoptic locus of the evolute of the parabola

$$y^2=4ax$$

The tangential equation of the evolute is $4a\xi^3=27\eta^2$

Making this homogeneous, we obtain—

$$4a\xi^3+27\eta^2(\xi x+\eta y)=4a\xi^3+27x\xi\eta^2+27y\eta^3=0.$$

Putting $-\xi/\eta=m$, we get

$$4am^3+27xm-27y=0 \qquad \dots \quad (1)$$

Putting $m=-m^{-1}$ in (1) we obtain

$$27ym^3+27xm^2+4a=0 \qquad \dots \quad (2)$$

Now, the eliminant of (1) and (2) will give the required orthoptic locus.

Ex. 5. The orthoptic locus of the evolute of $x^2/a^2+y^2/b^2=1$

is $$(a^2+b^2)(x^2+y^2)(a^2y^2+b^2x^2)^2=(a^2-b^2)^2(a^2y^2-b^2x^2)^2$$

a sextic curve.

144. If the equation of the curve be given in the parametric form $x=f_1(t)$, $y=f_2(t)$, we may proceed to find the orthoptic locus as follows:

Let t and t' be the parameters for any two points P and Q on the curve.

Then the equations of the tangents at P and Q may be written in the forms—

$$x+\phi(t)y=\psi(t) \quad \text{and} \quad x+\phi(t')y=\psi(t') \qquad \dots \quad (1)$$

where ϕ and ψ are functions of t.

The slopes of these tangents are—

$$-\frac{1}{\phi(t)} \text{ and } -\frac{1}{\phi(t')}.$$

Hence, if they are perpendicular, we must have

$$\phi(t).\phi(t')+1=0 \qquad \dots \quad (2)$$

If now we make the substitution $u=t+t'$ and $v=tt'$, then, by means of equations (1) and (2) we can express (x, y) the co-ordinates of the point of intersection of (1) in terms of a parameter u or v.

Ex. 1. Consider the curve $x=at^2$, $y=2at$.

Tangents at two points whose parameters are t and t' are

$$x-ty+at^2=0$$

$$x-t'y+at'^2=0$$

Since they are perpendicular, $tt'+1=0$, whence we get—

$$\frac{x}{-at}=\frac{y}{a(t^2-1)}=\frac{1}{t}.$$

i.e., $x+a=0$ is the orthoptic locus.

Ex. 2. Consider the parabola—

$$x=at^2+2bt+c\equiv\phi(t) \text{ (say)}$$

$$y=At^2+2Bt+C\equiv\psi(t) \text{ (say)}$$

The equation of the tangent at any point is—

$$x\psi'(t)-y\phi'(t)=\phi(t)\psi\,(t)-\phi\ .t)\psi$$

Here, $\phi(t)=at^2+2bt+c$

$\therefore\ \phi'(t)=2at+2b$

Similarly, $\psi'(t)=2At+2B.$

$\therefore$ The equation of the tangent is—

$$x(2At+2B)-y(2at+2b)=(Ab-aB)t^2+(Ca-cA)t+(Bc-Cb).$$

$$=f(t)\ \text{(say)} \qquad \ldots \quad (1$$

Similarly, the tangent at another point (t') is—

$$x(2At'+2B)-y(2at'+2b)=f(t'). \qquad \ldots \quad (2)$$

Since the tangents are perpendicular, we have—

$$\frac{\psi'(t)}{\phi'(t)}\cdot\frac{\psi'(t')}{\phi'(t')}+1=0 \qquad \ldots \quad (3)$$

If now we put $u=t+t'$, $v=tt'$, then, by means of equations (1), (2) and (3) we can find x and y in terms of u or v.

145. Equation of the Orthoptic Locus when the Polar Equation of a Curve is given:

Let $$r=f(\theta) \qquad \ldots \quad (1)$$

be the polar equation of a curve. Then the polar equation of its first positive pedal can be obtained, as usual, in the form

$$r=\phi(\alpha) \qquad \ldots \quad (2)$$

In this latter equation r is evidently the length of the perpendicular drawn from the pole on the tangent, and α is the angle which this perpendicular makes with the initial line.

Writing the equation of a tangent in the form—

$$x\cos\alpha+y\sin\alpha=p,$$

we have from equation (2)

$$x\cos\alpha+y\sin\alpha=\phi(\alpha) \qquad \ldots \quad (3)$$

A perpendicular tangent makes with the initial line an angle

$$(2k+1)\frac{\pi}{2}+\alpha$$

and consequently, its equation can be written as—

$$-x \sin \alpha + y \cos \alpha = (-1)^k \phi\{(2k+1)\frac{\pi}{2}+\alpha\} \qquad \dots \quad (4)$$

From equations (3) and (4) we can express x and y in terms of α.

Ex. 1. Find the orthoptic locus of the curve $r^m = a^m \cos m\theta$.

The equation of the pedal is—

$$r^{\frac{m}{m+1}} = a^{\frac{m}{m+1}} \cos \frac{m}{m+1}\theta.$$

or,
$$r = a\left\{\cos\frac{m}{m+1}\theta\right\}^{\frac{m+1}{m}}$$

Hence, the equation of any tangent may be written as—

$$x \cos \alpha + y \sin \alpha = a\left\{\cos\frac{m}{m+1}\theta\right\}^{\frac{m+1}{m}}$$

whence, proceeding as in the article, the equation of the orthoptic locus can be found.

Note. Theorems of § 142 also hold for the orthoptic loci.

Ex. 2. Find the orthoptic locus of the Cardioid $r = a(1+\cos\theta)$.

[The locus consists of a circle and a limaçon.]

CHAPTER VII

CHARACTERISTICS OF CURVES

146. Plücker's Equations:

We shall denote the degree of a curve	by n
,, class ,, ,,	,, m
,, number of nodes	,, δ
,, ,, ,, cusps	,, κ
,, ,, ,, bitangents	,, τ
,, ,, ,, stationary ,,	,, ι
and the deficiency by p.	

The six quantities $n, m, \delta, \kappa, \tau, \iota$ are called Plücker's numbers or the *Characteristics* of the curve.

Then we have $m=n(n-1)-2\delta-3\kappa$. (§121) ... (1)

$$\iota=3n(n-2)-6\delta-8\kappa. \quad (\S 112) \quad \dots \quad (2)$$

The corresponding numbers for the reciprocal curve are obtained by interchanging n and m, τ and δ, ι and κ.

Thus, from the reciprocal curve, we obtain—

$$n=m(m-1)-2\tau-3\iota \quad \dots \quad (3)$$

$$\kappa=3m(m-2)-6\tau-8\iota \quad \dots \quad (4)$$

From equations (1), (2), (3) and (4), we may express δ in terms of m, τ, ι, and τ in terms of n, δ, κ:—

Thus, $2\tau=n(n-2)(n^2-9)-2(n^2-n-6)(2\delta+3\kappa)$

$$+4\delta(\delta-1)+12\delta\kappa+9\kappa(\kappa-1) \quad \dots \quad (5)$$

$$2\delta=m(m-2)(m^2-9)-2(m^2-m-6)(2\tau+3\iota)$$

$$+4\tau(\tau-1)+12\tau\iota+9\iota(\iota-1) \quad \dots \quad (6)$$

The six equations $1-6$ are called *Plücker's Equations.**

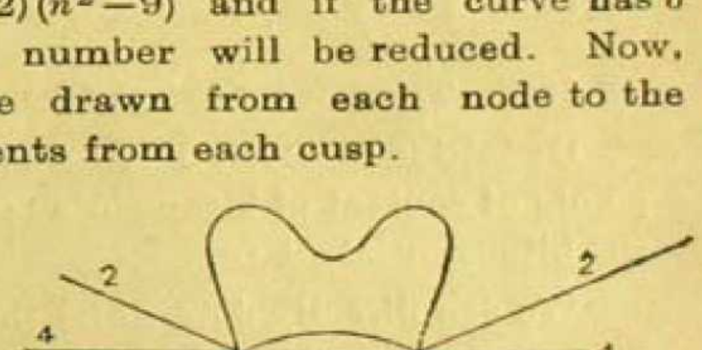
It is readily seen that a curve has the same characteristics as any projection of the curve, but curves with the same characteristics are not necessarily the projections of each other. All curves with the same characteristics, however, are said to be of the same *type*.

147. The above formula (5) is complicated in form and is not geometrically intelligible. We give here the following simpler form to the equation (5), which can be geometrically interpreted.

It can easily be shown that *two* bi-tangents coincide with each of the tangents drawn from a node to a curve, *three* coincide with each tangent drawn from a cusp, *four* coincide with each line joining two nodes, *six* coincide with each line joining a node to a cusp, and *nine* coincide with each line joining two cusps.

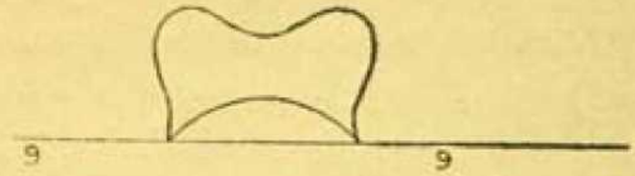
Now for a non-singular curve, the number of bitangents is given by $\tau=\frac{1}{2}n(n-2)(n^2-9)$ and if the curve has δ nodes and κ cusps, this number will be reduced. Now, $(m-4)$ tangents can be drawn from each node to the curve, and $(m-3)$ tangents from each cusp.

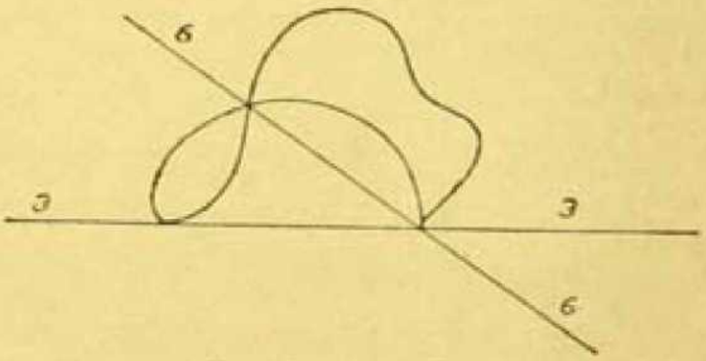
$\therefore$ $2\delta(m-4)$ bi-tangents coincide with the tangents drawn from the nodes, and $3\kappa(m-3)$ tangents coincide with those drawn from the cusps.



The number of lines joining the nodes is $\frac{1}{2}\delta(\delta-1)$,

* Plücker—*Solution d'une question foundamental concernant la théorié genérale des Courbes*—Crelle, Bd. 12 (1834), pp. 105-108. See also Cayley— Crelle, Bd. 34 (1847), p. 30.





and consequently they are equivalent to $2\delta(\delta-1)$ bitangents. The number of lines joining the nodes with the cusps is $\delta\kappa$, and they are equivalent to $6\delta\kappa$ bitangents. Finally, there are $\frac{1}{2}\kappa(\kappa-1)$ lines joining the cusps, and they are equivalent to $\frac{9}{2}\kappa(\kappa-1)$ bitangents. Thus the number of bitangents to a curve of order n, class m, with δ nodes and κ cusps is—

$$\tau=\tfrac{1}{2}n(n-2)(n^2-9)-2\delta(m-4)-3\kappa(m-3)-2\delta(\delta-1)$$
$$-6\delta\kappa-\tfrac{9}{2}\kappa(\kappa-1) \quad \ldots \quad (5)$$

This equation is, in fact, equivalent to (5). A similar expression may be obtained for the equation (6).

148. The Bitangential Curve: *

Definition: The curve which passes through the points of contact of bitangents of a given curve is called the bitangential curve.

We can directly determine the number of bitangents of a non-singular *n-ic* with the help of the bitangential curve, which intersects the *n-ic* in the points of contact of its bitangents. The order of a bitangential curve for a non-singular curve is, in general, $(n-2)(n^2-9)$.

* Prof. Cayley first determined the curve passing through the points of contact of bi-tangents—Crelle, Bd. 34 (1847), p. 37. Another method for determining this curve has been given by Salmon—Quarterly Journal of Mathematics, Vol. III, p. 317, and demonstrated by Cayley—Phil. Transactions (1859), p. 193, and (1861), p. 357.

The roots in λ/μ of the equation (1) of § 63 give the points where the line joining the points (x', y', z') and (x'', y'', z'') meet the curve $F=0$. We have seen in that article that if (x', y', z') lies on the curve and (x'', y'', z'') is a point on the tangent, $F(x', y', z') \equiv F'=0$ and $\Delta F'=0$. If the tangent at (x', y', z') touches the curve elsewhere, then, making $F'=0$ and $\Delta F'=0$ in that equation, the reduced equation of order $(n-2)$ must have equal roots. Consequently, the discriminant D of this reduced equation must vanish for (x', y', z') and (x'', y'', z''). But, as in the case of a point of inflexion $\Delta F'=0$ and $\Delta^2 F'=0$, and also $\Delta^2 F'$ contains $\Delta F'$ as a factor, in the case of a bitangent the discriminant D must contain $\Delta F'=0$ as a factor, and the condition thus obtained is the condition that the point (x', y', z') shall be a point of contact of a bitangent.

Now, the reduced equation takes the form—

$$\frac{1}{2}\lambda^{n-2}\Delta^2 F' + \frac{1}{3!}\lambda^{n-3}\mu\Delta^3 F' + \quad \ldots \quad + \mu^{n-2}F''=0$$

The discriminant D of this contains terms of the form $(\Delta^2 F')^{n-3}F''^{n-3}$, and therefore, D is of order $(n+2)(n-3)$ in (x'', y'', z''), of order $(n-2)(n-3)$ in (x', y', z'), and of order $2(n-3)$ in the co-efficients of the original equation.

But all the intersections of $D=0$ and $\Delta F'=0$ will coincide with (x', y', z'). For, the equation of the tangents drawn from (x', y', z') (§ 67) is of the form $k\Delta F' + D(\Delta^2 F')^2=0$. Hence, these tangents are intersected by $\Delta F'=0$ in no other point than (x', y', z'). Thus, if we put $\Delta F'=0$ in this equation, we see that $\Delta F'$ can neither meet D nor $\Delta^2 F'$ in any other point than (x', y', z').

Now, therefore, we have two curves $\Delta F'=0$ and $D=0$ of orders 1 and $(n+2)(n-3)$ respectively in (x'', y'', z'') and of orders $(n-1)$ and $(n-2)(n-3)$ in (x', y', z'), and the $(n+2)(n-3)$ points of intersection of the two curves all

coincide with (x', y', z'). Then by a known lemma,* the condition that the curves have other common points is of order

$$(n-2)(n-3)+(n-1)(n+2)(n-3)-(n+2)(n-3)$$

i.e., of order $\quad (n-2)(n^2-9) \quad$ in (x', y', z').

This condition $\wedge=0$, is therefore, of order $(n+2)(n-3)$ in the co-efficients of $\triangle F'$, of the first order in the co-efficients of D, and consequently, of order $(n+4)(n-3)$ in the co-efficients of the original equation.

Hence, the points of contact (x', y', z') of the bitangents of the curve $F=0$ are the points where $\wedge=0$ meets it, and their number is therefore $n(n-2)(n^2-9)$.

But there are two of these points on each bitangent, the number of bitangents is, therefore, $\frac{1}{2}n(n-2)(n^2-9)$.

Salmon has given an expression of the bitangential curve $\wedge=0$ for a general curve of order n.†

149. From Plücker's formulæ various other important results can be deduced:

From (3) and (4), by eliminating τ we obtain

$$\kappa-3n=\iota-3m, \quad \text{or,} \quad \kappa-\iota=3(n-m) \qquad \dots \quad (7)$$

The same equation also follows from (1) and (2) by eliminating δ. We see, therefore, that the four equations are not independent, but they are equivalent to three equations only.

From (1) and (3) it follows by subtraction that

$$n^2-m^2=2(\delta-\tau)+3(\kappa-\iota)$$

or,

$$n^2-m^2=2(\delta-\tau)+9(n-m)$$

$$\therefore \quad (n-m)(n+m-9)=2(\delta-\tau) \qquad (8)$$

* Salmon—H. P. Curves, § 381.

† Salmon—H. P. Curves, §§ 384-392.

From (5) and (8), we obtain—

$$\tfrac{1}{2}[n^2-m^2+3(n-m)]=\tfrac{1}{2}[2(\delta-\tau)+4(\kappa-\iota)]$$

$$=\delta-\tau+2(\kappa-\iota)$$

or, $$\tfrac{1}{2}n(n+3)-\delta-2\kappa=\tfrac{1}{2}m(m+3)-\tau-2\iota \quad \ldots \quad (9)$$

150. The equation (9) has a very simple geometrical interpretation :—

We have seen (§ 21) that a curve is uniquely determined by $\frac{1}{2}n(n+3)$ given points, or, in other words, the equation of a curve of the nth degree can be made to satisfy $\frac{1}{2}n(n+3)$ conditions.

But the existence of a node reduces the conditions by *one*, and that of a cusp by *two*. Therefore, the number of points determining a curve of the nth degree, with δ nodes and κ cusps, is $\frac{1}{2}n(n+3)-\delta-2\kappa$, and the above equation says that this number is equal to $\frac{1}{2}m(m+3)-\tau-2\iota$, which is the equivalent expression for the reciprocal curve.

Hence, *a curve and its reciprocal polar are determined by the same number of conditions*, as is otherwise evident, since when a curve is given, its reciprocal is determined (see § 62).

Again we have—

$$\tfrac{1}{2}\{n^2-m^2-3(n-m)\}=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}(m-1)(m-2).$$

The left-hand side, by (7) and (8), is equivalent to

$$\tfrac{1}{2}[2(\delta-\tau)+2(\kappa-\iota)]=\delta-\tau+\kappa-\iota$$

$$\therefore \quad \tfrac{1}{2}(n-1)(n-2)-\delta-\kappa=\tfrac{1}{2}(m-1)(m-2)-\tau-\iota=p \qquad (10)$$

The number p is called the Deficiency of the curve.

The equation (10) says that *a curve and its reciprocal have the same deficiency*. By introducing the number p

we can write the formulæ in the following simple forms:—

$$2(p-1)=\begin{cases} m+\kappa-2n \\ n+\iota-2m \\ n(n-3)-2(\delta+\kappa) \\ m(m-3)-2(\tau+\iota) \end{cases}$$

Prof. Cayley * puts $\iota+3n = \kappa+3m = a$

Then all the Plücker's numbers can be expressed in terms of three only, namely, n, m, a.

Thus, $\kappa=a-3m$, $\iota=a-3n$

$2\delta=n^2-n+8m-3a$, $2\tau=m^2-m+8n-3a$.

151. The Point and Line Deficiencies:

We have seen that a curve does not, in general, possess singular points, unless certain conditions are satisfied by the constants in the equation. But the general equation represents a curve which ordinarily possesses certain double or stationary tangents. Thus double tangents and stationary tangents may be reckoned as the ordinary singularities of a curve whose point-equation is given, while all other higher multiple tangents may be regarded as extraordinary singularities, the presence of which requires certain conditions to be fulfilled by the constants in the equation. But, if the tangential equation of a curve is given, the curve ordinarily possesses double and stationary points and cusps, but no singular tangents. Hence, double and stationary points are ordinary singularities of curves given by its tangential equation, but the presence of higher singular points are subject to certain conditions. Therefore, these ordinary singularities are such, that if any curve possesses the one, its reciprocal will possess the reciprocal singularitiy.

* Cayley—Quarterly Journal, Vol. XI, p. 185.

From all these considerations we are led to conclude that, if a curve has its maximum number of double points for a curve of that order, it has also the maximum number of double lines for a curve of that class. But it does not mean that the presence of double points on one leads to the presence of double lines on the other. The presence of maximum number of double points on a curve reduces its class to such an extent that the possible number of double tangents is thereby diminished, and made the same as the actual number. Similarly, in a curve of given class, the existence of the maximum number of double tangents reduces the order to such an extent that the possible number of double points is made the same as the actual number. Thus it is seen that for a curve of given order and class, the point-deficiency and line-deficiency are the same.

152. Curves with the same Deficiency:

THEOREM: *If two curves have a one-to-one correspondence, i.e., are so related that to any point of one corresponds a single point or tangent of the other, they have the same deficiency.**

Let S and S′ be any two curves of orders n and n' respectively, whose classes are m and m'. Let δ and δ' be the number of nodes and κ, κ' the number of cusps on them. Let p and p' be their deficiencies respectively.

Let A and A′ be any two fixed points in the plane, and P and P′ two corresponding points on S, S′ respectively. Let Θ be the locus of the intersection of the lines AP and A′P′.

The degree of Θ may be determined as follows:—

Consider any line AT through A. The number of intersections of AT with Θ will determine its degree. Now AT

* This proof was simultaneously given by Zeuthen (Compt. rend Ac. Sc., Paris, Vol. 52, 1869, p. 742) and Bertini (Giorn. di Mat., Vol. 7, 1869, pp. 105-106).

intersects S in n points, corresponding to which there are n points on S$'$. Consequently, the n lines which join these points to A$'$ intersect AT in n points which lie on Θ. Again, the point A is a multiple point of order n' on Θ; for AA$'$ intersects S$'$ in n' points which correspond to n' points on S. The n' lines joining these n' points on S to A meet AA$'$ in n' points coincident at A, which is a point on AT. Thus the degree of Θ is $n+n'$.

We obtain the same result by considering the intersections with Θ of any line through A$'$. The degree can, however, be easily determined by Chasles' Correspondence Formula, which will be explained in a subsequent Chapter.

The class of Θ is $2n'+m+\kappa-l$ or $2n+m'+\kappa'-l$, where l represents the number of cusps on S corresponding to l cusps on S$'$.

Consider the tangents to Θ which can be drawn from any point A. Since A is a multiple point of order n', the tangents at A are to be regarded as equivalent to $2n'$ among the tangents which can be drawn from A to the locus Θ. Other tangents may be determined in the following manner:

Any line AP will be a tangent to Θ when two of the lines A$'$P$'$ corresponding to AP coincide, without, at the same time, two of the lines AP corresponding to A$'$P$'$ coinciding; for in this latter case, the intersections of AP and A$'$P$'$ will be a node on the locus, and AP will not be a tangent in the ordinary sense.

The following three possible cases will then arise:

(1) AP touches the curve S;

(2) AP passes through a node on S;

(3) AP passes through a cusp on S$'$.

In case (1) AP will evidently also touch the locus Θ.

In case (2), according as the node on S corresponds to a node or a pair of distinct points on S′, we have, corresponding on the locus Θ, a node or a pair of distinct points; but in neither case is AP an ordinary tangent to Θ.

In case (3), according as the cusp on S corresponds to a cusp or a pair of coincident points on S′, AP passes through a cusp on the locus Θ, or else is an ordinary tangent.

Thus ordinary tangents are obtained only in cases (1) and (3).

Now the class of S being m, the m tangents to S will be the tangents to Θ drawn from A.

Again, if there are l corresponding cusps on S and S′, the number of tangents to Θ in case (3) will be $\kappa - l$. It is to be noted that there are l corresponding cusps on Θ. Hence, the total number of tangents, which can be drawn from A to the locus Θ, is $2n' + m + \kappa - l$,

i.e., the class of Θ is $2n' + m + \kappa - l$.

Similarly, considering the number of tangents which can be drawn to Θ from A′, we find for its class

$$2n + m' + \kappa' - l.$$

$$\therefore \quad 2n' + m + \kappa - l = 2n + m' + \kappa' - l,$$

or,
$$m + \kappa - 2n = m' + k' - 2n'.$$

But $m = n(n-1) - 2\delta - 3\kappa$ and $m' = n'(n'-1) - 2\delta' - 3\kappa'$.

$$\therefore \quad n(n-1) - 2\delta - 2\kappa - 2n = n'(n'-1) - 2\delta' - 2\kappa' - 2n'$$

or, $$2\left\{\frac{(n-1)(n-2)}{2} - \delta - \kappa\right\} = 2\left\{\frac{(n'-1)(n'-2)}{2} - \delta' - \kappa'\right\}$$

i.e.,
$$p = p'.$$

Cor.: The deficiency of a curve and its reciprocal polar is the same.

Ex. 1. The deficiency of a curve and its evolute is the same.

Ex. 2. The Hessian, the Steinerian and the Cayleyan of a curve have the same deficiency.

Ex. 3. A curve has the same deficiency as its inverse and pedal curves.

153. Extension of Plücker's Equations:

From what has been said in § 46 about multiple points on a curve, it follows that if the curve has multiple points, the equations 1–6 still hold, subject to certain equivalent conditions. In fact, Plücker's equations are still satisfied, if a multiple point of order k be regarded as equivalent to $\frac{1}{2}k(k-1)$ double points (nodes), and reciprocally, a k-ple tangent be replaced by $\frac{1}{2}k(k-1)$ bi-tangents.

Thus, if the curve has multiple points of orders k_1, k_2 ... we have

$$m=n(n-1)-2\delta-3\kappa-\Sigma k(k-1)$$

and
$$\iota=3n(n-2)-6\delta-8\kappa-3\Sigma k(k-1)$$

Reciprocally,

$$n=m(m-1)-2\tau-3\iota-\Sigma k(k-1)$$

$$\kappa=3m(m-2)-6r-8\iota-3\Sigma k(k-1)$$

where Σ extends over all the multiple points and tangents.

It can easily be deduced that the deficiency of a curve with only ordinary multiple points with distinct tangents is

$$\tfrac{1}{2}(m-2n+2).$$

For, in this case $\kappa=0$,

$$\therefore\quad p=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\Sigma k(k-1) \qquad (§\ 53)$$

or,
$$2p=(n-1)(n-2)-\Sigma k(k-1)$$

$$=\{n(n-1)-\Sigma k(k-1)\}-2(n-1)$$

$$=m-2n+2.$$

$$\therefore\quad p=\tfrac{1}{2}(m-2n+2).$$

Ex. Prove directly that the first polar of a point meets the curve $k(k-1)$ times at a k-ple point.

154. The Characteristics of the Hessian:

We have already seen (§ 90) that the order of the Hessian is $3(n-2)$. If the original curve has no singular points, the Hessian ordinarily has no double points,* and its Plückerian characteristics are easily found to be—

$$n'=3(n-2), \qquad \delta'=0, \qquad k'=0,$$

$$m'=3(n-2)(3n-7)$$

$$\tau'=\tfrac{27}{2}(n-1)(n-2)(n-3)(3n-8)$$

and $\quad \iota'=9(n-2)(3n-8) \qquad p'=\frac{1}{2}(3n-7)(3n-8)$

where n', m', δ', κ'...etc., denote the Plücker's numbers of the Hessian.

If, however, the original curve has nodes and cusps, each node is a node on the Hessian and each cusp is a triple point. Hence, these numbers must be modified accordingly, if the original curve has nodes and cusps.

155. The Characteristics of the Steinerian: †

There is a (1, 1) correspondence between the Steinerian and the Hessian. Hence, the deficiencies of the two curves must be the same. We have already found (§ 96) that the class of the Steinerian of a non-singular n-ic is $3(n-1)(n-2)$ and its order is $3(n-2)^2$.

A point will be a node or cusp on the Steinerian, if it is a point whose first polar has two nodes or two cusps. The number of first polars having a pair of nodes (§ 98) is—

$$\tfrac{3}{2}(n-2)(n-3)(3n^2-9n-5)$$

and the number having two cusps is $12(n-2)(n-3)$.

* Pezzo—Napoli, Rend, Vol. 22 (1883). Multiple points on the Hessian have been studied by T. R. Holcroft—Bull. of the American Mathematical Soc., Vol. 33, p. 90 (1927).

† Steiner—"Allgemeine, etc.", Crelle, Bd., 47, p. 1.

Hence, the characteristics of the Steinerian of a non-singular curve are—

$$n'=3(n-2)^2 \qquad m'=3(n-1)(n-2),$$
$$\delta'=\tfrac{3}{2}(n-2)(n-3)(3n^2-9n-5)$$
$$\kappa'=12(n-2)(n-3)$$
$$\tau'=\tfrac{3}{2}(n-2)(n-3)(3n^2-3n-8)$$
$$\iota'=3(n-2)(4n-9) \qquad p'=\tfrac{1}{2}(3n-7)(3n-8).$$

156. The Characteristics of the Cayleyan:*

This curve has evidently a (1, 1) correspondence with the Hessian and with the Steinerian, and has, therefore, the same deficiency.

We have already determined the class of this curve (§101) which also touches the inflexional tangents of the original curve. It has no inflexions, in general, and thus we obtain the following characteristics of the Cayleyan:

$$n'=3(n-2)(5n-11) \qquad m'=3(n-1)(n-2)$$
$$\delta'=\tfrac{3}{2}(n-2)(5n-13)(5n^2-19n+16)$$
$$\kappa'=18(n-2)(2n-5), \qquad \iota'=0$$
$$\tau'=\tfrac{9}{2}(n-2)^2(n^2-2n-1), \qquad p'=\tfrac{1}{2}(3n-7)(3n-8)$$

157. The Characteristics of the Inverse Curve:

From what has been said with regard to the process of inversion (§15) and the properties of inverse curves (§139), it follows at once that a curve and its inverse have a one-to-one correspondence and consequently, the characteristics of the latter can be easily determined.

If $$f\equiv u_0+u_1+u_2+\ldots+u_n=0 \qquad \ldots \quad (1)$$

be the equation of a curve, that of its inverse † is

$$u_0(x^2+y^2)^n+k^2u_1(x^2+y^2)^{n-1}+\ldots+k^{2n}u_n=0 \quad \ldots \quad (2)$$

* Clebsch—" Ueber einige von Steiner behandelte Curven "—Crelle's Journal, Bd., 64, pp. 288—93.

† A. S. Hart—Camb. and Dublin Math. Journal, Vol. VII (1)(1853).

Hence, the inverse curve has a multiple point of order n at each of the circular points I and J, and has also an *n-ple* point at the origin.

The degree of the inverse curve (2) is evidently $2n$. But if the origin is a *k-ple* point on (1), the degree of the inverse (2) is $2n-k$. The degree will be further reduced, if u_n, u_{n-1}, etc., contain some power of r as a factor, *i.e.*, if the curve (1) has multiple points at I and J.

Then, as before, $\qquad n'=2n.$

But, if the origin is a multiple point of order k, the degree of the inverse will be $2n-k$.

A node on the given curve inverts into a node on the inverse and in addition, each of the points I and J and the origin is an *n-ple* point on the inverse. Each of these three points is then equivalent to $\frac{1}{2}n(n-1)$ nodes.

Hence, $\qquad \delta'=\delta+\frac{3}{2}n(n-1).$

Again, a cusp inverts into a cusp, so that we have $\kappa'=\kappa$.

From these, the other characteristics of the inverse can be easily calculated.

Thus,
$$\begin{aligned} m' &= n'(n'-1)-2\delta'-3\kappa' \\ &= 2n(2n-1)-2\delta-3n(n-1)-3\kappa \\ &= \{n(n-1)-2\delta-3\kappa\}+2n \\ &= m+2n \\ \iota' &= 3n'(n'-2)-6\delta'-8\kappa' \\ &= 6n(2n-2)-6\delta-9n(n-1)-8\kappa \\ &= 3n(n-2)+3n \\ &= \iota+3n \end{aligned}$$

Similarly, $\qquad \tau'=2n\,(2n-7)+4mn+2\tau$

and $\qquad p'=p.$

It is to be noted, however, that in these investigations, the curve is supposed to have only nodes and cusps and no other higher multiple points.

It will be shown later on that the foci of a curve invert into the foci of the inverse curve, and that, if the origin is a focus on the curve, the circular points are cusps on the inverse.

If I and J are each a multiple point of order f, and the line at infinity a multiple tangent of order g, and f', g' denote reciprocal singularities, then the above results have to be modified.

Thus, $$n'=2n-2f-g'$$

$$m'=m+2n-2(2f+g')-(2f'+g)$$

$$\delta'=\tfrac{1}{2}(n-2f)(n-2f-1)+(n-f-g')(n-f-g'-1)+\delta$$

and so on.

Ex. 1. A $2n$-*ic* with n-*ple* points at I and J inverts into a curve of the same type.

Ex. 2. Show that a bitangent inverts into a circle having double contact with the inverse. Hence, find the number of circles passing through a given point and having double contact with a curve.

Ex. 3. Prove that an inflexional tangent inverts into a circle of curvature of the inverse. Find the number of circles of curvature of a given curve which pass through a given point.

Ex. 4. Shew that the curve

$$C(x^2+y^2)^2+2(lx+my)(x^2+y^2)+ax^2+2hxy+by^2+2gx+2fy=0$$

inverts into a cubic through the points I and J.

Ex. 5. Prove that through any point O on the above curve, three real circles of curvature pass, besides the circle of curvature at O, and the three points of osculation lie on a circle through O.

158. The Characteristics of the Pedal:

From what has been said in articles 137-39, it follows that the first positive pedal is the inverse of the polar reciprocal curve. Hence, the characteristics of the pedal can be obtained from those of the polar reciprocal curve by using the results of the preceding article, *i.e.*,

in the results of that article, we have simply to interchange n and m, δ and τ, ι and κ. Thus,

$$n'=2m,^{*} \qquad m'=n+2m, \qquad \delta=\tfrac{3}{2}m(m-1)+\tau$$

$$\kappa'=\iota, \qquad \tau'=2m(2m-7)+4mn+2\delta,$$

$$\iota'=3m+\kappa, \qquad p'=p.$$

In case of higher singularities, these numbers require to be modified according to the nature of the singularity.

If f, g, f', g', etc., have their significance as in § 157, we have—

$$n'=2m-f'-g, \qquad m'=n+2m-2(g+2f')-(2f+g')$$

and so on.

Taylor defines the pedal† of a pair of curves as follows:—

The locus of the vertex of a right angle whose arms envelop two curves of class m and class n respectively may be called the pedal of the two curves, or of the one *w.r.t.* the other, and the corresponding locus generated by the vertex of any other constant angle may be called a *skew* pedal. When one of the curves in the former case degenerates into a point, we obtain the ordinary pedal of the curve.

* Taylor says that the lines OI and OJ may be regarded as perpendicular to every one of the m tangents of a given curve of class m, which can be drawn from I and J respectively.

Each of I and J, therefore, is an *m-ple* point on the pedal and this having no other point at infinity is of order $2m$. When O is at a focus each of the lines OI and OJ is a tangent and also perpendicular to itself. Hence, these lines making up the point-circle at O belong to the locus and the remaining factor is of order $2(n-1)$.

Taylor—Messenger of Math., Vol. 16 (1887), p. 4.

† Dr. C. Taylor—Proc. of the Royal Soc. of London, Vol. 37 (1884), p. 139.

Ex. 1. Find the characteristics of the second, third, etc., pedals of a given curve.

Ex. 2. Find the characteristics of the first negative pedal.

[This may be regarded as the polar reciprocal of the inverse curve with respect to any point; and hence, the characteristics may be obtained from the results of §157.

$$n' = m + 2n, \quad m' = 2n$$

$$\delta' = 2n(2n-7) + 4mn + 2\tau, \text{ etc.}]$$

Ex. 3. Find the characteristics of the locus of the centre of a circle passing through a given point and touching a given curve.

Ex. 4. Find the characteristics of the envelope of a circle which passes through a given point and whose centre moves on a curve.

159. The Characteristics of the Evolute:

In order to determine the degree of the evolute, it is sufficient if we examine the number of points in which the line at infinity meets the evolute.

Now, the points at infinity on the evolute arise (1) from the points at infinity on the curve, (2) from the existence of points of inflexion on the curve.

Corresponding to a point at infinity on the curve, we have a cusp on the evolute, with the line at infinity as the cuspidal tangent.

Let M be any point on the line IJ, and M′ its harmonic conjugate, then the normal at M is the line IJ (§ 127). But if the consecutive points of the curve, antecedent and subsequent to M be L and N, their normals are LM′, NM′. Hence, M′ is a point through which three consecutive normals, *i.e.*, three consecutive tangents to the evolute pass, and is, therefore, a cusp with IJ for its tangent.

Now, the cuspidal tangent meets a curve in three consecutive points at a cusp, and the n points at infinity of the given curve give rise to the same number of cusps

* Steiner—"Über algebraische Curven und Flächen" Crelle, Bd., 49, p. 340.

on the evolute, which are then met by the line at infinity in $3n$ points.

Again, a point at infinity on the evolute is the point of intersection of two consecutive normals to the given curve which are parallel. The corresponding tangents to the given curve will therefore coincide, and the point of contact will be an inflexion on the given curve. Therefore, the ι points of inflexion on the given curve give rise to ι points at infinity on the evolute.

Hence, the line at infinity intersects the evolute in $\iota+3n$ points, or in other words, *the degree of the evolute is* $\iota+3n$.

If the curve passes through either I or J, these give rise to no points at infinity on the evolute, and consequently, the degree will be diminished by 3.

If, again, the line at infinity IJ touches the curve, the normals for the two consecutive points in which it meets the curve coincide with IJ, and consequently two consecutive tangents to the evolute coincide, *i.e.*, there is a point of inflexion on the evolute, having IJ for its tangent. But this takes the place of two cusps which we have when IJ meets the curve in distinct points, and the degree of the evolute is reduced by three.

Hence, if each of the circular points is an f-ple point on the curve and the line at infinity touches it at g points, the degree of the evolute is—

$$\begin{aligned} n' &= \iota+3n-3(2f+g) \\ &= a-3(2f+g) \end{aligned}$$

160. The Class of the Evolute :

The class of the evolute may be determined by considering the number of tangents which can be drawn to it from any point, or what is the same thing, by considering the number of normals which can be drawn from any point to the given curve. We may examine the case when the point is at infinity.

Now, the number of normals, distinct from the line at infinity itself, which can be drawn parallel to a given line, is equal to the class m of the curve. Again, the n normals, corresponding to the n points at infinity on the curve, coincide with the line at infinity, and consequently also pass through the assumed point.

Hence, *the number of normals which can be drawn to the given curve from any point is equal to the sum of the order and class of the curve, i.e., equal to the sum of the order of the curve and its reciprocal.*

Thus, the class of the evolute $m'=n+m$.

If the curve passes through a circular point, the normal at that point does not coincide with the line at infinity, and consequently, the number of normals is one less than in general. If either of these points be a multiple point of order f, the number will be reduced by f.

If, again, the line at infinity is a tangent to the curve, the number of finite tangents which can be drawn through a point at infinity is one less than in general, and consequently, the number of normals is also one less.

Thus, if the line at infinity touches the curve at g points and the curve has f-ple point at each of I and J, the number of normals is $m+n-2f-g$, *i.e.*, the class of the evolute is

$$m'=m+n-2f-g.$$

It is to be noticed that, in general, no two consecutive normals of the curve coincide. For, in that case, the corresponding tangents coincide with their normals and with each other ; and this can happen only in the exceptional case where the original curve has an inflexional tangent passing through I or J. Consequently, two consecutive tangents to the evolute cannot coincide, or in other words, *there is no inflexional tangents of the evolute.* Hence, $\iota'=0$.

If, however, the curve touches the line at infinity, there is a point of inflexion at infinity on the evolute,

and the curve passes through I or J, the evolute has an inflexional tangent passing through the same point.

Hence, in this case, $\iota'=2f+g$.

We can now easily calculate the other characteristics of the evolute by means of Plücker's formulæ.

Thus, $\kappa'=3n'+\iota'-3m'=3(\iota+3n)+\iota'-3(m+n)$.

Since, $\iota'=0$, $\kappa'=9n-3(m+n)+3\iota=3\{2n-m+\iota\}$

When the curve touches the line at infinity at g points and has f-ple point at each of I and J, we have—

$$\begin{aligned}\kappa'&=3n'+\iota'-3m'\\&=3\{\iota+3n-3(2f+g)\}+(2f+g)-3(m+n-2f-g)\\&=3a-3(m+n)-5(2f+g)\end{aligned}$$

where $a=\iota+3n$

And $\delta'=\frac{1}{2}\{n'^2-n'+8m'-3a'\}$

$\tau'=\frac{1}{2}\{m'^2-m'+8n'-3a'\}$

where $a'=\iota'+3n'=3a-8(2f+g)$.

Finally, $p'=\frac{1}{2}\{n'+\iota'-2m'\}+1$ (§ 150)

$=\frac{1}{2}\{\iota+3n-2(m+n)\}+1$

$=p$

Ex. 1. Find the characteristics of the evolute of $a^3y^2=x^5$.

$$n'=8, \qquad m'=8, \qquad p=0.$$

Ex. 2. Find the number of points on a curve where the osculating circle has a contact of the third order.

[The existence of a cusp of the evolute not lying on the line at infinity indicates the coincidence of three consecutive tangents of the evolute, and consequently of the coincidence of three consecutive normals to the curve, or in other words, corresponding to such a cusp on the evolute, we have a point on the curve where three consecutive normals coincide, *i.e.*, the osculating circle has a contact of the third order. Thus the number of such points $=\kappa'-n$, since the n cusps on IJ do not give any such point.

$$\kappa'-n=5n-3m+3\iota.\ ]$$

Ex. 3. How many lines are normal to a curve at two points?

A bitangent on the evolute corresponds to such a normal to the curve.

[Hence, there are τ' such points. But the n normals corresponding to the n points at infinity on the curve coincide with the line at infinity. Therefore, excluding these $\frac{1}{2}n(n-1)$ normals, corresponding to these n points, which coincide with IJ, there are then

$$\tau' - \tfrac{1}{2}n(n-1) = \tfrac{1}{2}(m^2 + 2mn - 4m - \kappa)$$

points.]

Ex. 4. Find the number of normals common to two curves of orders n_1 and n_2, and of class m_1 and m_2 respectively.

[Evidently such normals are the common tangents of their evolutes, whose classes are $m_1 + n_1$ and $m_2 + n_2$.

Hence the required number $= (m_1 + n_1)(m_2 + n_2)$.]

Ex. 5. Find the Plücker's numbers for the curve

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$$

[This is the evolute of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.]

161. The Characteristics of Parallel Curves:

To determine the degree of the parallel curve, we put $k=0$ in the equation (§ 140), which does not affect the terms of the highest degree in the equation. The result of putting $k=0$ is, however, the original curves written twice together with the two sets of m tangents drawn from the circular points I and J to the curve.

Hence, $$n' = 2(m+n).$$

Again, the number of tangents which can be drawn parallel to any given line is double that which can be so drawn to the original curve, *i.e.*, $m' = 2m$.

To each inflexional tangent on the original correspond two on the parallel curve, and therefore $\iota' = 2\iota$.

From these we can easily calculate the other characteristics.

Thus, $$\begin{aligned}\kappa' &= \iota' + 3(m' - n') \\ &= 2\iota + 3\{2m - 2(m+n)\} \\ &= 2a\end{aligned}$$

and so on.

The parallel curve and the original curve have the same normal and the same evolute.

Ex. 1. Find the characteristics of a parallel curve, when the original curve touches the line at infinity, and the points I and J are multiple points on the curve.

Ex. 2. Find the characteristics of the parallel to a parabola.

Ex. 3. Interpret the tangential equation of the parallel to the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

[The parallel is $\qquad (\xi^2 + \eta^2)\zeta^2 = \{a\xi\eta \pm k(\xi^2 + \eta^2)\}^2$]

Ex. 4. Find the characteristics of the envelope of a family of circles whose centres lie on a given curve and which touch a given circle.

Ex. 5. Find the pedal of the parallel curve and show that it is the same as the locus of a point Q taken on the radius vector OP of the pedal of the curve, such that $PQ = \pm k$. (This locus is called the conchoid of the pedal.)

162. The Characteristics of the Orthoptic Locus : *

In order to determine the degree of the orthoptic locus, we have to examine its intersections with the line at infinity. The line at infinity meets the locus only at I and J and in no other point, and each of I and J is a multiple point of order $\frac{1}{2}m(m-1)$. For, m tangents can be drawn to a curve from each of the circular points I and J and these may be taken in pairs in $\frac{1}{2}m(m-1)$ ways, and every pair may be regarded as at right angles to one another. Each of the points I and J is, therefore, a multiple point of order $\frac{1}{2}m(m-1)$, and these are, in general, the only points at infinity.†

* Taylor—" On the Order of Orthoptic Loci "—Messenger of Math., Vol. XVI (1887), pp. 1-5.

† O. Zimmermann calls this curve " Orthogonale "—Crelle, Bd., 126 (1903), p. 183.

For, let A and B be two points on the line IJ, harmonic conjugates with respect to I and J. The two tangents from A and B intersecting at C are then perpendicular, and consequently, C lies on the orthoptic locus. If then A and B approach I, C also approaches I, while C(IJ, AB) remains harmonic.

If the tangents from A and B are not consecutive, C becomes the point of contact of either tangent.

Now proceeding to the limit, we see that to each pair of tangents to the given curve from I corresponds a branch of the orthoptic locus through I, and the tangent to this branch is harmonic conjugate to IJ with respect to these tangents. Since the class of the curve is m, the m tangents through I may be taken to constitute $\frac{1}{2}m(m-1)$ such pairs of tangents, and consequently, there is the same number $\frac{1}{2}m(m-1)$ of branches of the curve which pass through I; or in other words, each of the points I and J is a multiple point of order $\frac{1}{2}m(m-1)$. Further, it can be easily seen that there is no other point of the locus on IJ.

Hence, the degree of the orthoptic locus is—

$$n' = m(m-1).$$

163. Class of the Orthoptic Locus:

To find the class of the orthoptic locus, it is sufficient to find the number of tangents which can be drawn to the locus from J.

Since each of I and J is a multiple point of order $\frac{1}{2}m(m-1)$, there are $m(m-1)$ such tangents at J (§ 65) which are to be regarded as tangents drawn from J to the curve. In order to find the other tangents we proceed as follows:

In the figure of § 142, let PP′ pass through J; then QR passes through I, for P(QR, IJ) and P′(QR, IJ) are harmonic.

In the limit, Q, R become the points of contact of the perpendicular tangents PQ, PR; and the tangent at P to the orthoptic locus passes through J, while QR passes through I.

If a line through I intersects the given curve at Q and R, and the tangents at Q and R meet in O, let L be a point on IQR, such that O(QR, IL) is harmonic. Now consider the envelope of the line OL. Every tangent from J to this envelope (distinct from IJ) will be a tangent to the orthoptic locus. Hence, the number of tangents to the orthoptic locus drawn from J (points of contact not lying on IJ) is equal to the class of the envelope of OL, which is evidently equal to the number of its tangents drawn from I.

It is readily seen that OL cannot pass through I, unless IQR touches the given curve at Q but not at R, or *vice versa*.

In this case then IQR will touch the envelope at a point K, such that (IK, QR) is harmonic. Therefore, each tangent from I to the curve is an $(n-2)$-ple tangent to the envelope, and consequently, the class of the envelope is $m(n-2)$.

Thus, finally the class of the orthoptic locus is—

$$m' = m(m-1) + m(n-2) = m(m+n-3).$$

In order to determine the number of cusps, it is to be noticed that, since a cusp may be considered as arising from the coincidence of two consecutive points on the locus, the points of intersection of an inflexional tangent to the given curve with a perpendicular tangent are cusps on the orthoptic locus, and in general, there are no other cusps. But, there are m perpendicular tangents to each inflexional tangent of the given curve. Consequently, the number of cusps on the orthoptic locus is $\kappa' = \iota m$.

From these we can easily calculate the other characteristics of the orthoptic locus.

Thus, $\iota' = \kappa' - 3(n' - m')$

$$= \iota m - 3\{m(m-1) - m(m+n-3)\}$$

$$= m(\iota + 3n - 6)$$

$$a' = \iota' + 3n' = \iota m + 3(3m - mn - m) + 3m(m-1)$$

$$2\delta' = n'^2 - n' + 8m' - 3(\iota' + 3n')$$

$$= n'^2 - 10n' + 8m' - 3\iota'$$

$$= m^2(m-1)^2 - 10m(m-1) + 8m(m+n-3) - 3m(\iota + 3n - 6)$$

$$= m\{(m+1)(m-2)^2 + 2\tau\}$$

$$2\tau' = m'^2 - m' - 8n' - 3(\kappa' + 3m')$$

$$= m\{m(m+n)^2 - (6m^2 + 6mn + n^2) - m + 22 + 2\delta\}$$

The deficiency is given by—

$$2p' = m' + \kappa' - 2n' + 2 \quad (\S 135)$$

$$= m(m+n-3) + \iota - 2m(m-1) + 2$$

$$= (m-1)(m-2) + 2mp$$

$$\therefore \quad p' = \tfrac{1}{2}(m-1)(m-2) + mp.$$

164. There is no difficulty in seeing how these numbers are to be modified, if the original curve touches the line at infinity or passes through the circular points at infinity.

Thus, if the line at infinity touches the curve at g different points, we have—

$$n' = (m-g)(m-1) \qquad m' = (m-g)(m+n-3-g)$$

$$\kappa' = (m-g)\iota; \qquad \text{and so on.}$$

In order to obtain these results we recall the results of § 143. It will be noticed that if the curve touches the

line at infinity, the absolute term will not appear in its tangential equation, and the co-efficients in the eliminant are of degree $(m-1)$, and the degree of the orthoptic locus is, therefore, $(m-1)^2$.

Hence, the orthoptic locus of a circle is a circle, while that of a parabola is a straight line (the directrix).

If, however, the linear as well as the absolute term are absent, the line at infinity is a bitangent or a stationary tangent, and the co-efficients in the eliminant are of degree $(m-2)$, and the orthoptic locus is of order $(m-2)(m-1)$.

And generally, *if the line at infinity is a multiple tangent of order g, the degree of the orthoptic locus of a curve of the mth class is* $(m-g)(m-1)$.

In a similar manner, the class is found to be—

$$m' = (m-g)(m+n-3-g).$$

The tangential equation of the evolute of the parabola is

$$4a\xi^3 = 27\eta^2.$$

Here the linear and the absolute terms are absent, and the orthoptic locus is a parabola.

The line at infinity is a bi-tangent to the evolute of the ellipse, and hence the orthoptic locus is a sextic curve.

Dr. C. Taylor in a note, published in the Proc. of the Royal Society of London, Vol. 37 (1884), pp. 138-141, propounded a number of theorems on the isoptic and orthoptic loci of a curve, where he remarks that these had been verified by analytical methods in an unpublished paper by one Mr. J. S. Yeo. The theorems stated here are taken from the said note.

In a similar manner, the Plücker's numbers are to be modified when the curve passes through I and J, or has multiple points at those points.

Ex. 1. Show that the orthoptic locus is a circle, when the curve is a central conic (or a circle).

Ex. 2. Show that the orthoptic locus is a straight line, when the curve is a parabola.

Ex. 3. The orthoptic locus of a quartic curve of class 3, touching the line at infinity at two points dividing IJ harmonically, is a straight line.

Ex. 4. Find the characteristics of the orthoptic locus when the inflexional tangents of the curve pass through the circular points.

$$[n'=(m+1)(m-2), \qquad m'=m(m+n-4), \quad \kappa'=\iota m-4.]$$

Ex. 5. Find the characteristics of the orthoptic locus of the evolute of a curve.

[The locus is the same as the locus of intersections of perpendicular normals of the curve. Hence,

$$n'=(m-1)(m+n-2), \qquad m'=(m-1)(4m+\kappa-6), \text{ etc.}]$$

165. The Characteristics of an Isoptic Locus : *

Since each of the circular points is an $m(m-1)$-*ple* point on the isoptic locus, proceeding in a manner similar to that in §162, it is found that the degree of the isoptic locus is—

$$n'=2m(m-1).$$

To find the class, the tangents at I are regarded as $2m(m-1)$ tangents which can be drawn from I to the curve; and proceeding as in §163, it will be found that there are other $2m(n-1)$ tangents which can be drawn from I to the curve.

Thus, $\quad m'=2m(m-1)+2m(n-1)=2m(m+n-2)$

Also, $\qquad \kappa'=2m\iota.$

From these we may calculate the other characteristics.

* C. Taylor—"Note on a Theory of Orthoptic and Isoptic Loci"—Proc. of the Royal Soc. of London, Vol. 37 (1884), pp. 138—41.

Thus, $\delta' = m(2m-3)(m^2-m-1)+2m\tau.$

$$\tau' = m\{2m(m+n)^2-(8m^2+8mn+n^2)-2m+12+2\delta\}.$$

$$\iota' = 2m(3m+\kappa-3) \qquad p' = (m-1)^2+2mp.$$

Ex. 1. Find the characteristics of the envelope of a chord of a curve which subtends an angle of given magnitude at a given point.

Ex. 2. Prove that the envelope of a circle which passes through a fixed point and subtends a constant angle at another is a limaçon.

Ex. 3. The isoptic locus of a parabola consists of the line at infinity twice and a central conic (*Taylor*).

Ex. 4. Prove that the points of contact of the tangents drawn from the circular points to any curve are single points on the orthoptic locus, and double points on Isoptic locus (*Taylor*).

166. Other Derived Curves :

Besides the curves discussed in the preceding article, a good number of others might be derived from a given curve by means of various known processes. But separate discussions of such curves are not considered necessary. Properties of such curves may be discussed independently as occasion arises. We shall, however, conclude this chapter with a few examples, which the students are required to work out for themselves.

Ex. 1. If on the radius OP of a straight line, we measure off distances $PQ = \pm k$, the locus of Q is a curve having a node at O and a tacnode at infinity with the given line as tangent.

[The locus is called the Conchoid of Nicomedes.]

Ex. 2. Find the locus of a point Q taken on the radius vector OP to a circle through the point O such that $PQ = \pm k$.

[This locus is called Pascal's Limaçon.]

Ex. 3. Discuss the nature of the origin in Ex. 2.

Ex. 4. Any straight line OP intersects two given curves at P_1 and P_2. Find the locus of a point P such that $OP = OP_1 - OP_2$.

[This curve is called the Cissoid of the curves for the pole O.]

Ex. 5. Find the characteristics of the locus in Ex. 4.

[$n' = 2n_1n_2$, $m' = n_1m_2 + n_2m_1 + 2n_1n_2$, $\kappa' = n_1\kappa_2 + n_2\kappa_1$,

where n_1, m_1, κ_1, etc , and n_2, m_2, κ_2, etc., are the characteristics of the two given curves.]

Ex. 6. Find the locus of P and its characteristics, if in Ex. 5, $OP = k_1OP_1 + k_2OP_2$, where k_1, k_2 are constants.

Ex. 7. Show that the cissoid of two circles, one of which passes through the origin, is a quartic curve having a node at the origin and two nodes at the circular points (A bi-circular quartic).

Ex. 8. If on the radius vector OP of a curve, $PQ = \pm k$ is measured off, where k is a constant, the locus of Q is the cissoid of the given curve and the circle with centre O and radius k.

[This curve is called the conchoid of the curve. *Cf.* Ex. 1.]

Ex. 9. Find the conchoid for the conic given by the general equation of the second degree.

Ex. 10. Show that the characteristics of the locus of the centre of a circle orthogonal to a given circle and touching a given $2n$-*ic* with *n-ple* points at I and J are the same as those of the reciprocal to the given $2n$-*ic*.

CHAPTER VIII

Foci of Curves.

167. Plücker's Conception of Foci: It is shown in Treatises on Conic Sections* that the foci of conics are the points of intersection of the tangents which can be drawn to the conic from the two circular points at infinity. This conception of the foci of a conic has been extended by Plücker,† who gives the following definition of the foci of a curve in general:—

Foci of a curve are the points of intersection of the tangents drawn to the curve from the circular points at infinity.

Theorem: *A curve of the mth class has, in general,* m^2 *foci, of which only m are real.*

Since m tangents can, in general, be drawn to a curve of class m from any point, m tangents can be drawn to the curve from each of the two circular points at infinity. These tangents are all imaginary, and they intersect in m^2 points, which are the foci of the curve.

But m, and only m, of these points will be real, if the curve is real; for, if one of the tangents drawn from the circular point I be of the form $A+iB=0$ (A and B being linear functions of the co-ordinates), one of the tangents drawn from the other circular point J will be of the form $A-iB=0$, these two intersecting in the real point $A=0$, $B=0$. All the other J-tangents will be of the form $C-iD=0$, none of which can intersect the I-tangent $A+iB=0$ in a real point, unless $C/A=D/B$, in which case $A-iB=0$ and $C-iD=0$ become indentical.

* Salmon—Conics, § 258, p. 238.

† Plücker—Crelle, Vol. X (1832), pp. 84—91. Also Cayley—Coll. Papers, Vol. VI, p. 515.

Therefore, a real focus of the curve is the intersection of an I-tangent with its *conjugate* J-tangent, and hence the number of real foci is m. For example, a conic has four foci, of which only two are real.

168. In the preceding investigation, it is assumed that the points I and J have no special position with reference to the curve. But if the curve passes through, or has singularities at, these points, the number of foci must be determined by a special method.

Theorem: *If the line at infinity is a multiple tangent of order g,* i.e., *touches the curve at g points, a curve of the mth class has* $(m-g)$ *real foci.*

Let the line IJ touch the curve at g points A, B, C, etc., distinct from I and J. Then IJ is to be regarded as g of the tangents from I or J to the curve. Then the I-tangents are made up of the line IJ counted g times and $(m-g)$ other tangents. Similarly, the J-tangents consist of the line IJ counted g times and $(m-g)$ other tangents. Then the foci of the curve, which do not lie at infinity, are the $(m-g)^2$ intersections of the $(m-g)$ I-tangents with the $(m-g)$ J-tangents, and of those only $(m-g)$ are real.

Again, the point I counts as $g(m-g)$ foci,* for, it may be regarded as the point of intersection of g J-tangents (IJ) with the $(m-g)$ I-tangents. Similarly, the point J counts as $g(m-g)$ foci. Then again, the g I-tangents (IJ) intersect the g J-tangents in g^2 points, of which only g are real, and these are the g points of contact of IJ with the curve. Thus the foci of a curve, which lie at infinity, consist of $g(m-g)$ at each of I and J, and g^2 on IJ, of which only g are real.

* The foci of a curve are to be distinct from the circular points I and J. Therefore they are not to be counted as foci. There are different opinions as to the way of counting the foci that lie at infinity. The reader is referred to Prof. Cayley's paper—Coll. Works, Vol. VI, p. 515, and also to Salmon's Higher Plane Curves, § 138.

Hence, the foci of the curve are—(1) $(m-g)^2$ finite, (2) $g(m-g)$ at I, (3) $g(m-g)$ at J, (4) g^2 on IJ, the total number being $(m-g)^2+2g(m-g)+g^2=m^2$; of these $(m-g)$ are real at a finite distance, and g at infinity.

Ex. The parabola is touched by the line at infinity. Its class being 2, it must have two real foci, of which one is at a finite distance and one is at infinity, *i.e.*, at the point of contact with the line at infinity.

169. Theorem: *If a non-singular curve of the mth class passes through the circular points at infinity, the curve has* $(m-2)$ *real single foci, and one real double (singular) focus.*

Since from a point on the curve, not more than $(m-2)$ tangents (exclusive of the tangent at the point) can be drawn to the curve, when I, J are points on the curve, $(m-2)$ tangents can be drawn to the curve from each of the points I and J, exclusive of the tangents at these points. Thus the curve has $(m-2)^2$ finite foci, of which $(m-2)$ only are real.

The two tangents at I and J are the limiting positions of the four tangents which can be drawn from the imaginary points I′ and J′ in the neighbourhood of I and J respectively. These four tangents intersect in four points, of which two are real and two imaginary. But when I′ and J′ move up to coincidence with I and J, the two real points coincide into one and form a *double focus*. This point is not usually included among the ordinary foci, and is called a *singular focus*.

In fact, the four points of intersection coincide into one, which, therefore, should properly be called a *quadruple* focus, but if we regard this as a real focus, it must be considered as a *double* one. Thus, the intersection of the tangents at I and J is a real *double* focus.

The foci of the curve then consist of:—

(1) $(m-2)^2$ finite foci, (2) 4 foci at the intersection of the tangents at I and J, (3) $2(m-2)$ foci at the

intersections of the $(m-2)$ I-tangents with the tangent at J counted twice, (4) $2(m-2)$ foci at the intersections of $(m-2)$ J-tangents with the tangent at I.

The real foci are then $(m-2)$ single foci and one double focus, which is the singular focus.

Ex. The circle passes through I and J and its class is two. The centre is the real (double) singular focus, or a quadruple focus, considering the imaginary foci also.

Combining this with the theorem in the preceding article, we obtain the theorem:—

If a non-singular curve of the mth class passes through the circular points at infinity, and the line at infinity is a multiple tangent of order g, the curve has $(m-g-2)$ real single foci, one (double) singular focus, and g real foci at infinity.

170. Theorem: *If the circular points are nodes on a curve of class m, the curve has $m-4$ real single foci and two real (double) singular ones, which are the two real points of intersection of the nodal tangents at the circular points.*

When the points I and J are nodes on a curve, the number of tangents which can be drawn from each to the curve is $m-4$ (exclusive of the nodal tangents). Therefore, the number of real single foci is $m-4$. Again, any one of the tangents at I intersects its conjugate nodal tangent at J in a real point, which is a double focus, and it intersects the other nodal tangent at J in an imaginary point. Since there are two pairs of conjugate nodal tangents, there will be two real *singular* foci. Hence, the real foci of the curve are (1) $m-4$ single foci, (2) two (double) singular foci.

In general, if each of the points I and J is a multiple point of order k on the curve, it can be shown, in a similar way, that there are k^2 singular foci, each of which counts

as four, and of which only k are real; and since only $m-2k$ tangents can be drawn from each of I and J to the curve, there are $(m-2k)^2$ ordinary foci, of which $m-2k$ are real single foci.

If the line at infinity is a multiple tangent of order g, the number of real single foci is $m-g-2k$, and that of the real singular foci is k, and there are g real foci at infinity.

171. Theorem : *If the circular points are cusps or inflexional points on a curve, of class m, it has $m-3$ real single foci and one real triple focus, which is the intersection of the cuspidal (or inflexional) tangents at the circular points.*

From a cusp or a point of inflexion, $m-3$ tangents, in general can be drawn to the curve, distinct from the cuspidal (or inflexional) tangent. Therefore, from each of the points I and J $(m-3)$ tangents can be drawn, and consequently, the number of real single foci is $m-3$.

Each of the cuspidal (or inflexional) tangents at I and J counts as three tangents, and therefore, their point of intersection counts as *nine* intersections, of which only three are real and the point is regarded as a real triple focus. Thus, the real foci of the curve are : —(1) $m-3$ real single foci, (2) one triple focus.

When the line at infinity is a multiple tangent of order g, the number of real single foci is $m-g-3$, and there is a triple focus.

172. In all these investigations we have assumed that, except at the circular points, the curve possesses no other singularities. But, every line joining the circular points to a double point has a contact of the first order with the curve at the double point, which can, therefore, be regarded as satisfying Plücker's definition of a focus, in which no distinction is made between contact and tangency. If, therefore, a curve has δ nodes and κ cusps, exclusive of

the circular points, these points should be included in the number of foci, and in the above formulæ, giving real single foci, m should be replaced by $m+2\delta+3\kappa$.

173. The Co-ordinates of the Foci :

The co-ordinates of the foci of a curve can be determined by forming the equation of the tangents which can be drawn from the point I to the curve by the method of § 67. The equation thus obtained will be of the form :

$$P+iQ=0$$

and the corresponding equation for the point J will be—

$$P-iQ=0.$$

The intersections of these two systems of tangents will, therefore, be determined by the equations

$$P=0, \qquad Q=0.$$

Thus, if f_1, f_2 denote the first differential co-efficients of f with respect to x and y, and f_{11}, f_{12}, etc., denote the second differential co-efficients, then the equation of the system of tangents drawn from the point $(1, i, 0)$ to the curve $f=0$, is obtained by forming the discriminant of the equation.

$$\lambda^n f+\lambda^{n-1}(f_1+if_2)+\tfrac{1}{2}\lambda^{n-2}(f_{11}+2if_{12}-f_{22})+\ldots=0$$

The foci are then determined by equating the real and the imaginary parts of the discriminant separately to zero.

In actual practice, however, the following method is very convenient :

Let $F(\xi, \eta, \zeta)=0$ be the tangential equation of a curve of class m.

The condition that the circular line $x-x'+i(y-y')=0$ through a focus (x', y') should touch the curve is obtained by putting 1, i, $(x'+iy')$ for ξ, η, ζ respectively in the given equation $F(\xi, \eta, \zeta)=0$ of the curve.

Equating the real and imaginary parts of this separately to zero, the co-ordinates (x', y') of the foci are determined as the intersections of two loci.

Ex. 1. Find the foci of the conic defined by the general equation of the second degree.

The tangential equation of the conic is—

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta = 0.$$

Substituting $1, i, -(x'+iy')$ for ξ, η, ζ in this equation, and equating the real and imaginary parts separately to zero, we obtain for the locus of (x', y')—

$$C(x^2 - y^2) + 2Fy - 2Gx + A - B = 0$$

$$Cxy - Fx - Gy + H = 0$$

which represent two rectangular hyperbolas. From these equations the co-ordinates of the foci can be determined. The chord of contact is evidently the directrix.

By analogy, the chord of contact of the two circular lines through a focus of any curve may be called the corresponding directrix.

Ex. 2. Find the real foci of the curve given by—

$$4\xi^2\eta^2 - 3\xi^2 + 1 = 0$$

Making the equation homogeneous by introducing powers of ζ, we have—

$$4\xi^2\eta^2 - 3\xi^2\zeta^2 + \zeta^4 = 0$$

Now, putting $\xi = 1$, $\eta = i$ and $\zeta = -(x'+iy')$, the resulting equation gives us—

$$(x'+iy')^2 = 4 \quad \text{or,} \quad (x'+iy')^2 = -1$$

Hence, we have either $x'+iy' = \pm 2$ or $x'+iy' = \pm i$

The first solution gives $x' = \pm 2$, $y' = 0$

The second gives $x' = 0$, $y' = \pm 1$

$\therefore$ The foci are $(\pm 2, 0)$ and $(0, \pm 1)$.

Ex. 3. Find the real foci of the curve $\zeta^4 + \xi^2\eta^2 = 0$

Putting $\xi = 1$, $\eta = i$, $\zeta = -(x'+iy')$

we obtain $(x'+iy')^4 = 1$

$\therefore$ $(x'+iy')^2 = \pm 1$, whence $x'+iy' = \pm 1$, or, $x+iy' = \pm i$

which give the foci $(\pm 1, 0)$ and $(0, \pm 1,)$

Ex. 4. Find the foci of :—

$$(i)\ \zeta^3+\eta\zeta^2+2\xi\zeta^2-2\eta^2\zeta+2\xi\eta\zeta+2\xi^2\eta=0$$

[Foci $(1,\ \pm1)$, $(0,\ 1)$]

$$(ii)\ 27k\eta^2\zeta\ +\ 4\xi^3=0$$

[Focus $(-2/k,\ 0)$].

174. From what has been said above, it follows that if the tangential equation of a curve be given, the co-ordinates of the foci can be determined for special forms of the equation. Using tangential co-ordinates, let $\alpha, \beta, \gamma, \ldots$ denote the foci and ω, ω' the two circular points at infinity. Then, since the lines $\alpha\omega$, $\alpha\omega'$, $\beta\omega$, $\beta\omega'$,...are tangents to the curve, the tangential equation must be of the form $\alpha\beta\gamma\delta\ldots=\omega\omega'\phi$, where ϕ is an expression of order $m-2$ in tangential co-ordinates.

Now, for a curve of the second class, $\omega\omega'$ is constant and the equation becomes $\alpha\beta=\omega\omega'=$constant. The geometrical interpretation of this is that *the product of the perpendiculars drawn from the two foci on any tangent is constant.*

The equation of a curve of the third class can be similarly put into the form $\alpha\beta\gamma=\omega\omega'\delta$, and the geometrical interpretation of this equation is that *the three tangents drawn from the foci α, β, γ (besides the circular lines) are concurrent in a fourth point δ*, and therefore the product of the perpendiculars from the three foci of a curve of the third class on to a tangent is in a constant ratio to the perpendicular drawn on the same tangent from a fourth point δ. A similar interpretation can be given in the general case.

175. Equation of Confocal Curves :

Let $f\,(\xi,\ \eta,\ \zeta)=0$ be the tangential equation of a curve of class m. Then the equation of a curve having the same oci as the given curve may be written as

$$f(\xi,\ \eta,\ \zeta)+\omega\omega'\phi(\xi,\ \eta,\ \zeta)=0 \qquad \ldots \quad (1)$$

where $\omega\omega'$ is of degree 2 and represents the circular points at infinity, $\phi(\xi, \eta, \zeta)=0$ represents a curve of class $m-2$.

Considering the two circular points as constituting a degenerate curve of class 2, a curve confocal with the given curve must touch the $2m$ common tangents to $\omega\omega'=0$ and $f=0$. Hence, the tangential equation of a curve confocal with $f=0$ must contain—

$$\tfrac{1}{2}m(m+3)-2m=\tfrac{1}{2}m(m-1)$$

arbitrary co-efficients.

Since there are $\frac{1}{2}m(m-1)$ co-efficients in ϕ, the equation (1) represents a curve touching the common tangents of $f=0$ and $\omega\omega'=0$.

Cor: In the Cartesian system

$$f(\xi, \eta)=0 \text{ and } F(\xi, \eta)=0$$

are confocal, if $f(\xi, \eta)-F(\xi, \eta)=(\xi^2+\eta^2)\phi(\xi, \eta)$.

176. Determination of Singular Foci:

The singular foci of a curve are the intersections of the circular lines which are asymptotes of the curve. If then the asymptotes are found by the usual method, the singular foci can be easily determined.

Let $$(x-x')+i(y-y')=0$$

be an asymptote to the curve; then

$$(x-x')-i(y-y')=0$$

is also an asymptote. These two evidently intersect in the point (x', y') which is a *singular* focus of the curve.

Ex. 1. Find the singular focus of

$$2x(x^2+y^2)=a(3x^2-y^2)$$

Two circular asymptotes are $x+iy=a$, and $x-iy=a$ which evidently intersect at the point $(a, 0)$, which is a singular focus.

To find the ordinary focus, we consider the intersection of the line $y=ix+c$ with the curve.

By putting $y=ix+c$ in the equation of the curve, we find—

$$4x^2(ic-a)+2cx(c+ia)+ac^2=0$$

one root of which is evidently infinite, as it should be.

The other two roots will be equal,

if $$4c^2(c+ia)^2=16\ ac^2(ic-a)$$

or, if $$(c+ia)(c-3ia)=0$$

i.e., if $$c=-ia \quad \text{or,} \quad c=3ia$$

$c=-ia$ gives another infinite root, which gives the singular focus.

If $c=3ia$, the line $y=ix+3ia$ is a tangent, or, $(x+3a)+iy=0$ is a tangent. Hence $(x+3a)-iy=0$ is also a tangent. They intersect in the ocus $(-3a, 0)$.

Ex. 2. Find the foci of $x(x^2+y^2)=ay^2$

The line $y=ix+c$ intersects the curve in points given by—

$$x\{x^2+(ix+c)^2\}=a(ix+c)^2$$

or, $$x^2(a+2ic)+x(c^2-2iac)-ac^2=0 \qquad \dots \quad (1)$$

one point of intersection is evidently at infinity. The other two points are given by equation (1).

These will be equal, if $(c^2-2iac)^2+4ac^2(2ic+a)=0$

i.e., if $$c+4ai=0, \text{ whence } c=-4ai.$$

Hence, the ordinary focus is $(4a, 0)$. The singular focus will be obtained by making the co-efficient of x^2 vanish, *i.e.*, by making $2ic+a=0$.

Hence, the singular focus is $(-\frac{1}{2}a, 0)$.

Ex. 3. Show that the following two curves have no foci, singular or ordinary :

(*i*) A curve of class m touching the line at infinity at the circular points and $m-3$ other points.

(*ii*) A curve of class m touching the line at infinity at the circular points and $m-4$ other points, and having cusps at the circular points.

Ex. 4. Find the foci of the curve defined by $x=at^p$, $y=at^{p+q}$.

The lines $y=\pm ix+c$ meet the curve, where

$$at^{p+q}=\pm i(at^p)+c \qquad \dots \quad (1)$$

At a consecutive point, we have—

$$at^{p+q}+(p+q)at^{p+q-1}\partial t=\pm i\{at^p+apt^{p-1}\partial t\}+c \qquad \dots \quad (2)$$

From (1) and (2) we obtain $(p+q)at^{p+q-1} = \pm iapt^{p-1}$

i.e., $t^{q-2} = \pm \dfrac{ip}{p+q}$, or, $t = \left(\pm \dfrac{ip}{p+q}\right)^{\frac{1}{q-2}}$

Hence, the point of contact is given by—

$$x = a\left(\pm \frac{ip}{p+q}\right)^{\frac{p}{q-2}}, \quad y = a\left(\pm \frac{ip}{p+q}\right)^{\frac{p+q}{q-2}}$$

Substituting these in the equation of the lines, we obtain—

$$a\left(\pm \frac{ip}{p+q}\right)^{\frac{p+q}{q-2}} = \pm ia\left(\pm \frac{ip}{p+q}\right)^{\frac{p}{q-2}} + c$$

Putting $k = \pm \dfrac{ip}{p+q}$, we get $ak^{\frac{p+q}{q-2}} = \pm iak^{\frac{p}{q-2}} + c$

whence $c = ak^{\frac{p+q}{q-2}} \pm iak^{\frac{p}{q-2}}$

Hence, the lines are— $\left(y - ak^{\frac{p+q}{q-2}}\right) = \pm i\left(x \pm ak^{\frac{p}{q-2}}\right)$

$\therefore$ The foci are $\left(\pm ak^{\frac{p}{q-2}},\ ak^{\frac{p+q}{q-2}}\right)$

Ex. 5. Find the foci of the following curves :

(*i*) $(x^2+y^2)^2 - 8(x^2-y^2) + 15 = 0$

[Singular foci $(\pm 2, 0)$, ordinary foci $(\pm \sqrt{15}/2, 0)$]

(*ii*) $(x+y)(x^2+y^2) + 2x(x-y) = 0$

[Singular focus (0, 1), ordinary foci

$(-3 \mp 2\sqrt{\sqrt{2}+1},\quad - \mp 2\sqrt{\sqrt{2}-1})$]

177. A new Theory of Foci :*

If in the equation $F(1, i, -x-iy)=0$ of § 173 we put $z=x+iy$, F may be considered as a monogenic function of a complex variable z, and consequently, the theory of foci of algebraic curves reduces to the algebra of binary forms.

If then F breaks up into n factors of the forms—

$$(z-a_1-i\beta_1)(z-a_2-i\beta_2)\ldots(z-a_n-i\beta_n)=0 \qquad \ldots \quad (1)$$

it also breaks up into n other factors of the forms—

$$(z-a_1+i\beta_1)(z-a_2+i\beta_2)\ldots(z-a_n+i\beta_n)=0 \qquad \ldots \quad (2)$$

If now we take one factor of each along with one factor of the other, and put each equal to zero, then, since these can be taken in n^2 different groups, there are in all n^2 foci, of which n are real with co-ordinates

$$(a_1, \beta_1), (a_2, \beta_2), (a_3, \beta_3), \ldots (a_n, \beta_n)$$

and $n(n-1)$ are imaginary, as has already been shown. These latter foci are so arranged that they lie on the perpendiculars drawn at right angles to the lines joining the real foci through their middle points.

Writing the equation $F(\xi, \eta, \zeta)=0$ in the form—

$$\zeta^n - nP_1\zeta^{n-1} + \frac{n(n-1)}{2!} P_2\zeta^{n-2} - \ldots + (-1)^n P_n = 0 \quad (1)$$

where $P_1, P_2, \ldots P_n$ are functions of ξ, η, and putting $\zeta=0$, we find that

$$P_n \equiv p_0\xi^n + p_1\xi^{n-1}\eta + p_2\xi^{n-2}\eta^2 + \ldots + p_n\eta^n = 0. \ \ldots \quad (2)$$

* This method of treatment was given by Siebeck—"Ueber eine neue analytische Behandlungwise der Brennpunkte"—Crelle, Bd. 64, pp. 175—182. He has shown that from known properties of complex functions many interesting results on the properties of foci can easily be deduced with the help of this new method of treatment. See also Zimmermann—Crelle, Bd. 126 (1903), p. 171.

gives the tangents of the angles which the tangents drawn from the point $\zeta=0$ make with the axis of x.

If $a_1, a_2, \ldots a_n$ be the angles, we have—

$$\tan \Sigma a=(p_1-p_3+p_5-\ldots)\div(p_0-p_2+p_4-\ldots) \quad (3)$$

Again, the foci are obtained by putting $(1, i, -z)$ for ξ, η, ζ, so that the equation giving the foci becomes—

$$z^n-n\text{P}_1 z^{n-1}+\frac{n(n-1)}{2!}\text{P}_2 z^{n-2}+\ldots+(-1)^n\text{P}_n=0$$

where $\text{P}_n\equiv(p_0-p_2+p_4-\ldots)+i(p_1-p_3+p_5-\ldots)$

If $\beta_1, \beta_2, \ldots \beta_n$ be the angles which the radii to the foci make with the axis of x, we have—

$$\Pi r\cos\Sigma\beta=(-1)^n(p_0-p_2+p_4-\ldots)$$

$$\Pi r\sin\Sigma\beta=(-1)^n(p_1-p_3+p_5-\ldots)$$

$$\therefore\quad \tan\Sigma\beta=(p_1-p_3+p_5-\ldots)\div(p_0-p_2+p_4-\ldots)$$

$$=\tan\Sigma a$$

$$\therefore\quad \Sigma a=\Sigma\beta+n\pi.$$

Thus we obtain the theorem :—

The tangents drawn from any point to a curve make with a fixed line angles whose sum differs from the sum of the angles made by the lines joining the same point to the foci by a multiple of π.

Ex. 1. If $\text{P}_n=p_n(\cos\varpi_n+i\sin\varpi_n)$, p_n is the product of the distances of the n real foci from the origin, and ϖ_n is the sum of the angles which the radii to the foci make with the x-axis.

Ex. 2. Tangents are drawn from any point to two confocal curves. The tangents to one curve make with any fixed line angles whose sum differs from that of the angles made with the same line by tangents to the other curve by a multiple of π.

178. In finding the co-ordinates of the foci of a curve in § 173, we obtained two equations of the forms P=0, Q=0, which, when geometrically interpreted, will lead to very interesting results.

Writing the condition that the line

$$(x-x')+p(y-y')=0$$

should touch the curve in the form—

$$ap^n+bp^{n-1}+cp^{n-2}+\quad\ldots\quad=0$$

where a, b, c,...etc., are functions of x', y', it is evident that $-b/a$, c/a, etc., are the sum, the sum of products in pairs, etc., of the tangents of the angles, which the tangents to the curve through (x', y') make with the axis of x.

If now we put $p=i$ and equate the real and the imaginary parts to zero, we obtain the two equations:

$$\mathrm{P}\equiv a-c+e-\ldots\text{etc.}=0,\quad \mathrm{Q}\equiv b-d+f-\ldots\text{etc.}=0$$

Now, if α_1, α_2, α_3, ...be the angles which the tangents make with the axis of x, we have—

$$\tan\Sigma\alpha^* = \frac{b-d+f-\ldots}{a-c+e-\ldots}$$

Hence, from the first equation P=0, we obtain the theorem:

The sum of the angles made with the axis of x by the tangents through (x', y') is an odd multiple of $\frac{1}{2}\pi$.

Also the second equation expresses the fact that the sum is either zero or a multiple of π.

Thus, if the sum of the angles be given, *i.e.*, if $\Sigma\alpha=\theta$, we obtain the following theorem:

The locus of a point, such that the sum of the angles made with a fixed line by the tangents through it drawn to a curve of the nth class is given, is an n-ic.

* Hobson—Plane Trigonometry, § 49, p. 47.

If the fixed line is taken for the axis of x, the equation of the locus is found to be—

$$(a-c+e-\ldots)\tan\theta=(b-d+f-\ldots)$$

where $a, b, c, \ldots$ etc., are functions of order n in (x, y).

It follows then that the locus passes through the foci of the curve.

Ex. 1. Find the distances of the real foci from the origin.

Let $f(\xi, \eta)=0$ be the tangential equation of the curve, and (r, θ) the polar co-ordinates of a focus. Then $(x-r\cos\theta)\pm i(y-r\sin\theta)=0$ are tangents whose co-ordinates are—

$$\frac{1}{-re^{\pm i\theta}}, \quad \frac{\pm i}{-re^{\pm i\theta}}$$

Substituting these values for ξ, η in the given equation, we obtain—

$$f(-r^{-1}e^{-i\theta}, -ir^{-1}e^{-i\theta})=0$$

and

$$f(-r^{-1}e^{i\theta}, +ir^{-1}e^{i\theta})=0$$

Eliminating θ between these equations, we obtain an equation giving the values of r.

Ex. 2. Prove that every focus of a curve is also a focus of its evolute. [Tangents through one of the circular points coincide with the normal.]

179. Foci of Inverse Curves:

If a curve be inverted from any point, the inverse points of the foci of the original curve are the foci of the inverse curve.

A focus of a curve has been defined as an indefinitely small circle which has double contact with the curve.

Now, if C is a circle which has double contact with a curve at two points P and Q, and if the origin of inversion be not on the circle, the inverse of the circle C will

be another circle having double contact with the inverse curve at the inverse points P′ and Q′. If further C be a point-circle, its inverse will also be a point-circle. Hence, the focus of a curve is inverted into an indefinitely small circle having double contact with the inverse curve, or in other words, the inverse of a focus is a focus of the in-inverse curve.

The inverses of the lines joining J to the intersections of the curve with CI are tangents at I to the inverse curve. Hence, when C is a focus of the curve, two tangents at I to the inverse curve coincide, and we obtain the theorem :—

The inverse of a curve with respect to a focus has cusps at the circular points.

For instance, the inverse of a conic with respect to one of its real foci is a limaçon, which is a quartic curve having cusps at the circular points and a node at the origin (focus).

Ex. 1. The inverse of any curve with respect to a singular focus O has also a singular focus at O.

Ex. 2. A limaçon is self-inverse with respect to the circle through the node with its centre at the ordinary focus.

[The ordinary focus of the limaçon $r = a + b\cos\theta$, with node at the pole, is

$$\{(b^2 - a^2)/2b,\ 0\}$$

The equation of the curve with the ordinary focus as pole is—

$$4b^2r^2 - 4br(a^2\cos\theta + b^2) + (a^2 - b^2)^2 = 0]$$

180. Reciprocal with respect to a Focus:

The reciprocal of a curve with respect to a focus is a curve through the circular points, and the directrix reciprocates into a singular focus.

If we reciprocate with respect to the focus O, *i.e.*, a circle with O as centre, the lines OI and OJ reciprocate into the points I and J; and consequently the points I, J are on the reciprocal curve, *i.e.*, the reciprocal passes through the circular points.

The directrix is the chord of contact of the tangents OI and OJ, whose points of contact reciprocate into the tangents at I and J. Hence, the reciprocal of the directrix is the point of intersection of these tangents, which is a singular focus.

It will be noticed that the reciprocals, with respect to any point O, of the foci are the lines joining the intersections of the reciprocal curve with OI and OJ.

Ex. The reciprocal of a conic with respect to a focus is a circle, the directrix reciprocating into the centre of the circle.

181. Foci of Circular Curves:

*The locus of the singular focus of a circular curve of order n, passing through $\frac{1}{2}n(n+3)-3$ other points, is a circle.**

Since $\frac{1}{2}n(n+3)-1$ points in all are given, an infinity of curves can be drawn through them, any particular member being determined by an additional condition. If then we are given a point consecutive to I, *i.e.*, if the tangent at I is given, the curve is determined, and consequently its tangent at J also. Thus, if F is a singular focus, *i.e.*, the intersection of the tangents at I and J, when IF is given, JF is determined. Hence, the singular focus F is the intersection of the corresponding rays of two homographic pencils, whose vertices are I and J. Consequently the locus of F is a conic through I and J, and is therefore a circle.

Cor: The locus of the centres of a coaxal system of circles is a straight line. For, the circle which is the locus of the singular foci breaks up into two right lines when to IJ corresponds JI for the other pencil, and IJ cannot be a tangent to the circle.

* Salmon—Higher Plane Curves, § 145.

Ex. 1. Prove that the locus of the focus of a curve of class m, having given the line at infinity for an $(m-1)$-ple tangent and $2m-1$ other tangents, is a circle.

[To be given that IJ is an $(m-1)$-ple tangent is equivalent to $\frac{1}{2}m(m-1)$ tangents, and thus $\frac{1}{2}m(m-1)+(2m-1)=\frac{1}{2}m(m+3)-1$ tangents are given.]

Ex. 2. Being given three tangents to a parabola, the locus of the focus is a circle.

Ex. 3. The locus of the focus will be a circle for a curve of the third class, having the line at infinity for a bitangent and five other tangents.

Ex. 4. If four foci of a curve are concyclic, there are three other sets of four concyclic foci.

[The pencil of four tangents from I has the same cross-ratio as the pencil of four tangents from J, and the ratio remains unchanged by two interchanges in the order.]

Ex. 5. Prove that the coaxal family of circles through two foci of a curve have two other foci as limiting points.

[Given two real foci A, A′ of a curve, the lines AI, AJ ; A′I, A′J meet in two imaginary points B, B′ which are also foci of the curve ; the relation between the two pairs of points is that the lines AA′, BB′ bisect each other at right angles. Any circle through A, A′ cuts any circle through B, B′ at right angles. The points B, B′ are called *antipoints*.]

CHAPTER IX

TRACING OF CURVES

Section I : Approximate Forms of Curves :

182. Analytical Triangle : *

In order to determine approximate shapes of curves, defined by given equations containing a finite number of terms, near particular points at a finite or at an infinite distance, a geometrical method, which consists in representing the equation on the *Analytical Triangle*, is often very useful. This triangle is a modification of Newton's parallelogram, which was an arrangement of squares, like those on a chess-board, each square being allotted to one term of the general equation of any degree. The fact that all the terms of a complete equation of any degree are contained in squares which occupy half of Newton's Parallelogram, led De Gua to replace the parallelogram by a triangle containing one more square on each side than the degree of the equation.

183. First Approximation :

An equation is said to be *placed* upon the triangle by making a definite mark (*) in the centre of each square, which corresponds to a term of the equation. The use of the triangle as an analyser lies in the fact that, if the different marks representing the terms be joined, so as to form a polygon, no mark lying outside it, then the locus of the equation, formed by retaining only the terms which correspond to any side, is

* For a complete discussion the student is referred to Frost's Curve Tracing, Chap. IX, pp. 130-45.

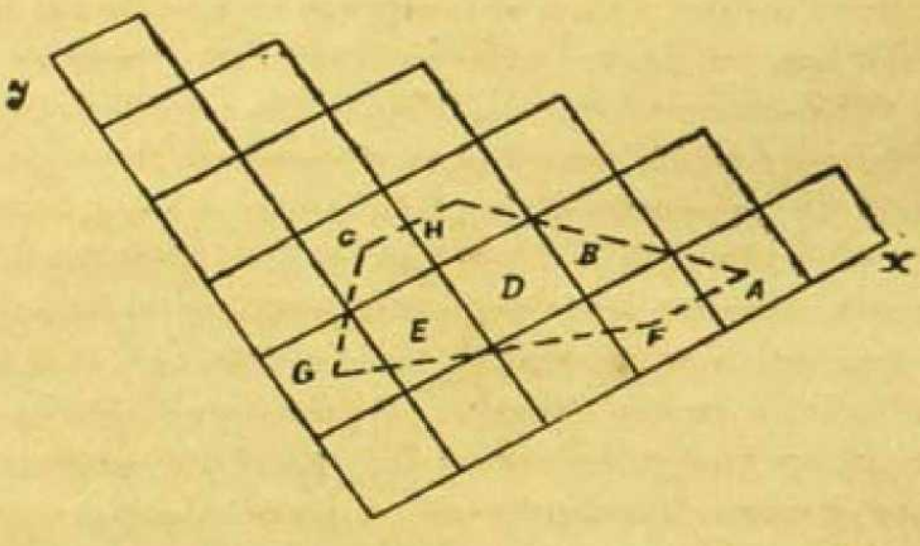
one or more simple parabolic curves or right lines, each of which is a *first* approximation to the form of the curve—

either (1) at an infinite distance, if *all* the rejected marks lie on the same side of the line as the right angle of the triangle,

or, (2) near the origin, if they lie on the opposite side, in which case the equation contains no constant term.

The case when the sides of the polygon are parallel to, or coincident with, the sides of the triangle is to be excepted. Thus the equation—

$$ax^4 + bx^3y + cxy^2 + dx^2y + exy + fx^3 + gy + hx^2y^2 = 0$$



is placed on the triangle, as shown in the figure. A, B, C, D, E, F, G, H represent the different terms of the equation, so that we have the polygon ABHCGF, no mark lying outside it.

The equation corresponding to the side CG is—

$$cxy^2 + gy = 0 \qquad \text{or} \quad cxy + g = 0,$$

which represents a hyperbola to which the curve approximates at infinity, since all the rejected marks lie on the same side as the right angle; with this relation between x and y (being very great) every other term will vanish compared with the terms retained.

The equation corresponding to the side GF is $gy+fx^3=0$, and this represents a cubical parabola to which the curve approximates at the origin, since the rejected marks lie on the opposite side. With this relation between x and y, every other term vanishes, compared with any one of the terms retained, when x and y are very small.

Lastly, the side FA is parallel to Ox, and therefore it is excepted.

184. Practical Method:

The following method of considering the triangle is much more convenient in fixing more accurately the position of the marks which correspond to fractional indices.

Take a right-angled isosceles triangle, whose sides are in the directions of Ox and Oy. Divide the hypotenuse into as many equal parts as the degree of the equation, and through these points of section draw lines parallel to the sides of the triangle. The triangle is then divided into a number of smaller triangles, the vertices of which may be appropriated to the terms of the equation. If a fractional index occurs in the equation, its appropriate position is obtained by the intersection of the two lines parallel to Oy, Ox drawn through the corresponding points on the sides Ox and Oy respectively.

185. Properties of the Analytical Triangle: *

When all the terms of the equation of a curve are represented upon the triangle, the following properties hold:—

(1) If all the terms be rejected, except the terms which lie in the given line, the resulting equation gives one or more constant values of the ratio $y^q : x^p$, the values of p and q depending only on the direction of the line, and consequently the same for all parallel lines.

* See Frost's Curve Tracing—§§ 150, 151, pp. 132-34.

(2) When the line meets both sides of the triangle or those sides produced beyond the hypotenuse, with the relation y^q/x^p=constant, the terms of the original equation, corresponding to the marks which lie on the same side of the line as the right angle, will be less, and those corresponding to the marks on the opposite side will be greater, than any of the terms corresponding to the marks on the line, when x and y are indefinitely great. When x and y are indefinitely small, the reverse is the case, and there is no mark at the right angle.

(3) When the line is parallel to one side of the triangle, the equation obtained gives one or more right lines parallel to the other.

(4) If all the other terms be rejected, except those which lie on a side of the triangle, the resulting equation determines the points of intersection of the curve with the corresponding axis.

All these properties of the analytical triangle lead at once to the properties of the analytical polygon we have mentioned before. Thus it is seen that if any line gives the first approximation to an asymptote, the terms to be taken into account for the second approximation are obtained by moving the line parallel to itself, until it passes through another mark which corresponds to the additional term that has to be taken into account.

Thus, in the figure of §183, the side GF gives a first approximation, while the second approximation is obtained by moving the line GF parallel to itself until it passes through another mark E, and the form is given by—

$$exy+fx^3+gy=0, \text{ and so on.}$$

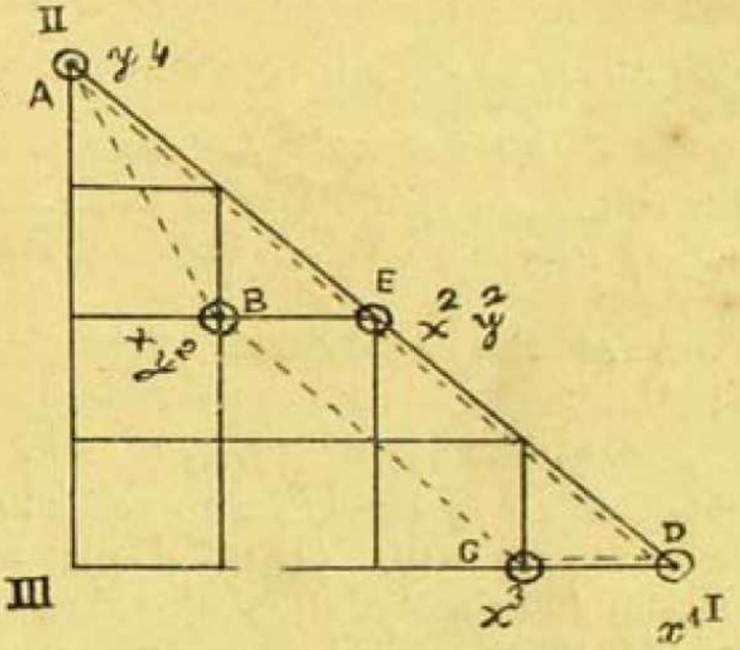
Ex. 1. Consider the curve $(y^2-x^2)^2+2axy^2-5ax^3=0$.

The homogeneous form of the equation is—

$$(y^2-x^2)^2+2axy^2z-5ax^3z=0.$$

Place the equation on the Analytical Triangle, as shown in the figure. The polygon is ABCDE.

The side AED gives at an infinite distance $(y^2-x^2)^2=0$, *i.e.*, the points at infinity on the curve are in the direction of the lines $y \pm x=0$ and these points are double points on the curve.



The next approximation to the curve at infinity will be obtained by moving the line AED parallel to itself, until it passes through one other term, *i.e.*, it is given by

$$(y^2-x^2)^2-5ax^3=0 \qquad \text{or} \qquad y^2=x^2 \pm \sqrt{5a}x^{\frac{3}{2}}.$$

$$i.e., \; y = \pm x \pm \tfrac{1}{2}\sqrt{5ax.}$$

The side AB gives the shape at the origin (III), *i.e.*, $y^2+2ax=0$, which is a parabola.

The side BC gives the shape at the origin of another branch of the curve, which is given by $ax(2y^2-5x^2)=0$,

i.e., the origin is a triple point, the tangents at which are

$$x=0, \quad \sqrt{2}y \pm \sqrt{5}x=0.$$

The side CD gives for the points of intersection with the axis of x the equation $\qquad x^4-5ax^3=0$ or $x^3(x-5a)=0$,

which gives three coincident points at III and one point at $x=5a$.

The next approximation, at the point $x=5a$, is obtained by moving the line CD parallel to itself until it passes through another point E.

The equation thus obtained is $x^3(x-5a)+2axy^2-2x^2y^2=0$, which gives $y^2=\frac{25a}{8}(x-5a)$ at the point $x=5a$.

Ex. 2. Find the approximate shape at infinity of the curve—

$$x^3y^2+xy^5-y^7-x^7=0.$$

Ex. 3. Employ the analytical triangle to find the form of the branches of the curve $x^4+y^4-(x+y)x^2=0$, near the origin.

186. Use of the Analytical Triangle in Three Variables:

In the case of three variables we proceed as follows:—

The sides of the triangle, supposed equilateral, are divided each into as many (n) equal parts as the degree of the equation, and through the points of section lines are drawn parallel to the sides. In this way we obtain $\frac{1}{2}(n+1)(n+2)$ vertices of small triangles in which the fundamental triangle is divided, each of the vertices being associated with one term of the equation.

When an equation of the nth degree is to be placed on the triangle, the highest powers of the variables x, y, z are associated with the vertices of the original triangle. The other terms are arranged successively in order. This, in fact, is the same as given in the preceding article, since the Cartesian is a special system of homogeneous coordinates, when the third side is at infinity ; and the system is rectangular when the angle at the third vertex III is a right angle, *i.e.*, when we take a right-angled isosceles triangle, as has been done.

The aggregate of terms on a side gives the intersections of the curve with the corresponding side of the fundamental triangle.

If one vertex be free, the curve passes through that vertex, which may be considered as one of the intersections on the x axis or the y axis.

If the next following terms on both the sides be present, the aggregate of these terms gives the tangent at this vertex. If only one be present, one of the axes is a tangent.

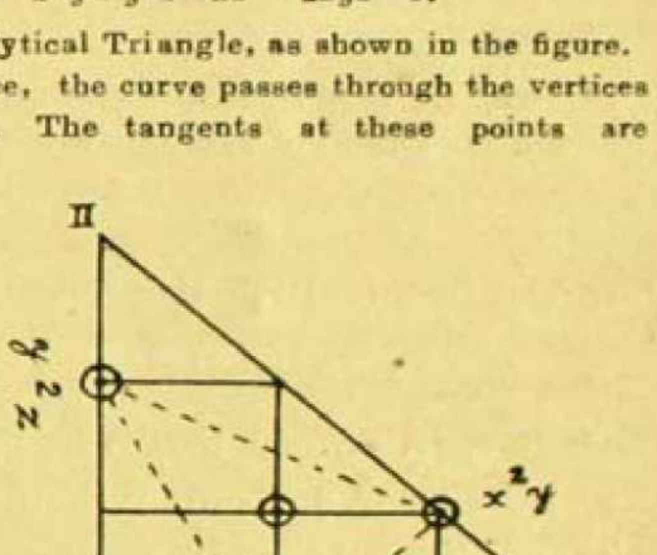
If both the next following terms are absent, the vertex is a double point on the curve, and the next three terms lying on a line give the tangents at the double point.

The method will be best illustrated by the following example:

Ex. Consider the equation $x^2y + y^2z - xz^2 - 2xyz = 0$.

Place the equation on the Analytical Triangle, as shown in the figure.

Since all the vertices are free, the curve passes through the vertices of the fundamental triangle. The tangents at these points are respectively $y=0$ (at I), $z=0$ (at II), $x=0$ (at III).



In order to determine approximate shapes of curves at any of the vertices, we must first put that co-ordinate equal to unity, whose highest power stands at the vertex. Now the first approximation to the curve at the vertex II is the tangent given by y^2z and y^2x, but since y^2x is absent the tangent is given by $y^2z=0$, *i.e.*, $z=0$, putting $y=1$.

To determine the approximate shape of the curve, we must turn the line about the point y^2z till it passes through another term x^2y, and the next approximation to the curve is $y^2z + x^2y = 0$, or $x^2 + z = 0$, obtained after putting $y=1$.

Thus the shape of the curve at II is that of the parabola $x^2 + z = 0$.

If we determine the three points in which this parabola intersects the curve by putting the equation in the form $y(yz + x^2) - xz(2y + z) = 0$, we see that two of these lie on the line $2y + z = 0$, while the third coincides with the vertex II.

In the same way, for the vertex III, we have for the next approximation the equation $y^2z - xz^2 = 0$, or $y^2 - x = 0$.

This is a parabola to which the curve approximates near the vertex III and the two finite points of intersection lie on $x=2z$.

For the approximation at the vertex I, we obtain the parabola $y=z^2$.

The two finite points of intersection of this parabola with the curve lie on the line $y=2x$.

187. Newton's Method of Approximation:

In order to determine the approximate forms of a curve near the origin, we may conveniently make use of the process given by Newton.*

When the curve passes through the origin, we have to transform the curve so that one of the axes may be a tangent, *i.e.*, if the origin is an ordinary point on the curve, we may assume the equation in the form

$$y=Ax+Bx^2+\dots$$

or, if it be a multiple point, the form to be assumed is

$$y=Ax^r+Bx^{r+r_1}+Cx^{r+r_1+r_2}+\dots$$

where r, r_1, r_2...are positive.

In order to determine the nature of the origin and the form of the curve in its neighbourhood, it is convenient to determine the first terms of this expansion. The form of the curve near the origin resembles the curve $y=Ax^r$.

In the equation of the curve, put $y=Ax^r$, and determine the positive quantity r by the condition that the indices of two or more terms shall be equal and less than the index of any other term. We may often do this by trial, by equating the indices of each pair of terms, and observing whether the resulting value of r is positive, and these equal indices are not greater than the indices of some other term.

* Newton—Methodus Fluxionum et Serierum infinitarum, etc., under the title De reductione affectarum equationum (opusc. ed., Castillon, Vol. I, p. 37) Newton gives the rule by means of a diagram of squares.

Thus, having found r, we may obtain the value of A by equating to zero the quantity multiplying the terms with equal indices, r and A being thus determined, the first approximation is given by $y = \mathrm{A}x^r$. The next approximation is obtained from $y = \mathrm{A}x^r + \mathrm{B}x^{r+r_1}$, by finding the values of r_1 and B by a similar process.

Ex. Consider the curve $x^4 - axy^2 + y^4 = 0$.

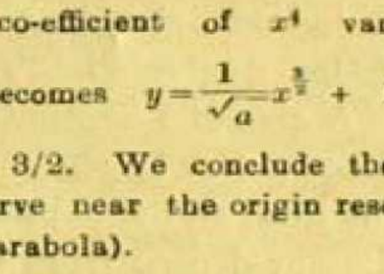
Evidently the origin is a triple point with the axes as tangents. Writing $y = \mathrm{A}x^r$ in the equation, it becomes $x^4 - a\mathrm{A}^2x^{2r+1} + \mathrm{A}^4x^{4r} = 0$.

First, let $4 = 2r + 1$, which gives $4r = 6$. Hence the equal indices are less than the index of the other term. The equation then becomes

$$x^4(1 - aA^2) + A^4x^6 = 0.$$

Now determine A so that the co-efficient of x^4 vanishes, *i.e.*, put $\mathrm{A} = 1/\sqrt{a}$. The equation then becomes $y = \dfrac{1}{\sqrt{a}}x^{\frac{3}{2}}$ + higher terms, whose indices are greater than 3/2. We conclude therefore that the form of one branch of the curve near the origin resembles that of the curve $ay^2 = x^3$ (semi-cubical parabola).

Next, make $4 = 4r$, *i.e.*, $r = 1$, and the remaining index is 3, which is less than the equal indices. We must, therefore, reject this case.



Finally, let us assume $2r + 1 = 4r$, or $r = \frac{1}{2}$, the equation becomes

$$x^4 - (a\mathrm{A}^2 - \mathrm{A}^4)x^2 = 0$$

The equal indices are less than the remaining index, and putting $\mathrm{A}^2a = \mathrm{A}^4$, we have $\mathrm{A}^2 = a$, or, $\mathrm{A} = \sqrt{a}$

Thus the equation becomes $y = \sqrt{ax}$, *i.e.*, $y^2 = ax$, which is a parabola. Hence another branch of the curve resembles a parabola at the origin.

Since it is symmetrical about the axis of x, the form of the curve is shown as in the figure. (The vertical line representing the x-axis.)

188. Application of Newton's Method in Three Variables:

The above method is equally applicable to the study of approximate forms of curves in the neighbourhood of the vertices of the fundamental triangle, when the curve is defined by its homogeneous equation. In this case we have to put one of the variables equal to unity.

In fact, what we have said above is really equivalent to finding the points of intersection of $y=\mathrm{A}x^r$ with the given curve and then determining those which lie very near the origin. For this purpose, we have to consider those terms for which the index is small as compared with those of the remaining terms. Thus, from the term $x^p y^q$ we obtain $\mathrm{A}^q x^{p+qr}$.

We have now to put $p+qr=\lambda$, where λ is less than any of the indices, since for small values of x, higher powers must be neglected. The terms which have this equal index λ give the geometrical representation.

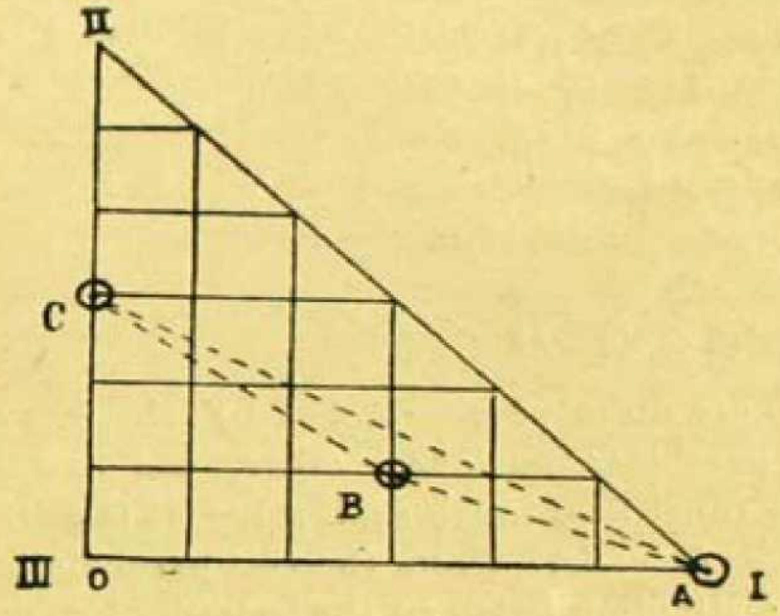
Since $p+qr$ is constant for all these terms, they lie on a right line, and λ is proportional to the length of the perpendicular drawn from the origin on to this line. Since λ is less than any other index, the line is nearest the origin, and the terms which lie along this line give an approximate form of the curve in the neighbourhood of the origin. Thus appears the truth of the statement in § 185.

189. Infinite Branches:

The method adopted by Newton can conveniently be applied for determining the infinite branches of the curve as well. In this case y is expanded in descending powers of x, *i.e.*, in the form $y=\mathrm{A}x^r+\mathrm{B}x^{r-r_1}+\mathrm{C}x^{r-r_1-r_2}+\ldots$ and the equal indices should be made greater than any of the remaining indices. In fact, in this case $p+qr=\lambda$, where λ is greater than any of the other indices, *i.e.*, we have to consider the terms which lie on a line furthest from the origin.

Ex. Consider the curve $x^6+2x^3y-y^3=0.$

Place the equation on the Analytical Triangle as shown in the figure. The sides of the triangle ABC determine the approximate forms of the curve. The curve passes through the origin (III) and the



vertex (II), and not through (I). The origin is a triple point. The shape at the origin is given by AB and BC. AB gives a point of inflexion and the side BC gives a semi-cubical parabola.

The side AC gives $x^6-y^3=0$, or, $x^2=y$, which is the form of the curve at infinity in the direction of the axis of y. The same result is obtained by applying Newton's method of approximation, namely, putting $y=Ax^r$ and proceeding as before. The next approximation is obtained by putting $y=Ax^r+Bx^{r-r_1}$, and so on.

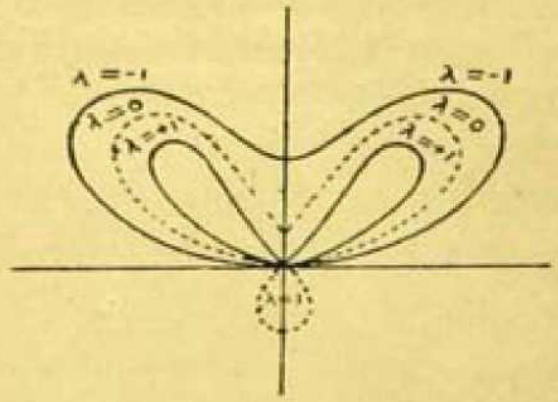
190. Branches with Higher Singularities:

Consider the system of curves $x^4+y^4-4x^2y+\lambda y^2=0$. None of these curves have real points at infinity.

For $x=0$, we obtain $\quad y^2(y^2+\lambda)=0.$

At the points $y=\pm\sqrt{-\lambda}$, they are symmetrical with respect to the horizontal tangent. Therefore, an essential distinction must be made between the curves for $\lambda>0$ and $\lambda<0$ between which $\lambda=0$ lies. The adjoining diagram gives an example of each.

For $y=0$, we have $x^4=0$. This point, however, cannot be a point of undulation, because for $x=0$, we have also $y^2=0$. As an approximation we obtain the aggregate of three terms



$$\lambda y^2-4x^2y+x^4=0,$$

which breaks into two parabolas

$$\lambda y=x^2(2\pm\sqrt{4-\lambda})$$

These are no more intersected by the curves. Since the two parabolas touch each other, the two branches of the curve must also touch each other at the origin. Hence we have a *tacnode*. From the two parabolas we can also see the distinction between the curves for positive and negative values of λ. Since for $\lambda<0$, we have $\sqrt{4-\lambda}>2$, the parabolas have external contact, one lying upward and the other downward (for instance $\lambda=-1$).

If $4>\lambda>0$, the parabolas have internal contact, one lying inside the other (for instance $\lambda=+1$).

If $\lambda=4$, the origin is the only real point of the curve, which now breaks up into two imaginary conics with four-pointic contact at the origin:

$$(x^2+iy^2-2y)(x^2-iy^2-2y)=0$$

If $\lambda>4$, the curve is wholly imaginary, when $\lambda=\pm\infty$, it reduces to the axis of x counted twice, whence for very great negative values of λ, the forms with external contact follow again.

We have further to consider the case $\lambda=0$. Here in the analytical polygon, the term y^2 is suppressed and we have the dotted periphery. We obtain as above a parabola $x^2-4y=0$ as an approximation for the branch, which touches the x-axis, and the parabola $y^3-4x^2=0$

for the other branch, which therefore is perpendicular to the first and has a cusp. In fact, each of the curves has a triple point at the origin with the tangents $x^2y=0$. If, however, we consider the horizontal tangent, then the curve is bounded by the two parabolas and the horizontal tangent, and we are in a position to obtain their shapes for each value of λ.

If, again, $\partial f/\partial x \equiv 4x^3-8xy=4x\,(x^2-2y)=0$,

we have besides $x=0$, as already found, points which lie on the parabola $x^2=2y$.

Putting this in the equation, $y=\sqrt{4}-\lambda$ will be positive for real contact, otherwise $x=\sqrt{2y}$ becomes imaginary.

The line $y=\sqrt{4-\lambda}$ is a bi-tangent of the curve ; and $y=-\sqrt{4-\lambda}$ is a bi-tangent with imaginary points of contact.

191. Determination of the Asymptotes:

The aggregate of terms on a side of the Analytical Triangle gives an equation which is obtained by putting the corresponding variable to zero. This equation determines the points of intersection of the corresponding side with the curve and gives the equations of the lines joining the opposite vertex with these points. When the third side is supposed to lie at infinity, these give the directions of the lines joining the origin to the points at infinity on the curve. It is then necessary to determine the tangents at these points, which are called the *asymptotes* of the curve.

If, for such a point at infinity, we put $y/x=m$, this gives also the direction of the asymptote, *i.e.*, an asymptote is of the form $y=mx+c$, where c is to be determined by the condition that it passes through another consecutive point at infinity.

For this purpose, we put $y=mx+c$ in the equation of the curve ; the resulting equation must have two roots infinite ; (or in the homogeneous form $y=mx+cz$, one value of $z=0$ and two values of $x=\infty$).

Ex. 1. Consider the cubic $y^3-x^2y+2y^2z+4yz^2+xz^2=0$ ... (1)

Putting $z=0$ in this equation, we obtain $y^3-x^2y=0$, which gives the directions of the lines joining the origin with the points at infinity on the curve. These are the lines $y=0$, $y\pm x=0$.

If we put $y=0$ in the equation (1), we see that all the terms vanish, except xz^2, which contains z^2 as a factor. Hence $y=0$ itself is an asymptote.

Next put $y=x+cz$ in (1). We have then—

$$(x+cz)^3-x^2(x+cz)+2z(x+cz)^2+4(x+cz)z^2+xz^2=0,$$

or, $$(2c+2)x^2z+(3c^2+4c+5)xz^2+(c^3+2c^2+4c)z^3=0 \qquad \ldots \quad (2)$$

This equation must have one root infinite, *i.e.*, we must have

$$2c+2=0 \quad \text{or} \quad c=-1,$$

and the asymptote is $y=x-1$.

Similarly, by putting $y=-x+c$, we obtain $c=-1$.

$\therefore$ The other asymptote is $y+x+1=0$

Ex. 2. Find the asymptotes of the following curves :

(*i*) $x^3+y^3=a^3$ (*ii*) $x^2y=x^3+x+y$

(*iii*) $(x+y)^2(x+2y+2)=x+9y-2$.

Ex. 3. Find the asymptotes of the curve—

$$y=x\frac{x^2+a^2}{x^2-a^2}.$$

192. Special Methods:

In some cases the asymptotes of a curve can be determined by a simple inspection of its equation. For instance, if the equation of the curve be written in the form

$$t_1.t_2.t_3 \ldots\ldots t_n + z^2\phi = 0,$$

where t_1, t_2.........are all linear functions of the variables, and ϕ is of degree $(n-2)$, then t_1, t_2, t_3,...t_n are the n

asymptotes of the curve and the points where they meet the curve again lie on the curve ϕ.

Thus we have the theorem: *The $n(n-2)$ points where the asymptotes of an n-ic meet the curve again lie on a curve of order $(n-2)$.*

The equation of a curve can always be put into the form—

$$(y-m_1x)(y-m_2x)\dots\dots\dots(y-m_nx)+u_{n-1}z$$
$$+u_{n-2}z^2+\dots\dots+u_0z^n=0$$

or, $(y-m_1x+c_1z)(y-m_2x+c_2z)\dots\dots(y-m_nx+c_nz)=z^2\phi.$

The terms of the nth degree in x and y are the same for both sides of the equation, and the n quantities c_1, $c_2 \dots c_n$ are determined by comparing the co-efficients of the terms of order $(n-1)$, which must be the same in both.

The following illustrations will explain the process:

Ex. 1. Consider the curve:

$$(x+2y)(3x+2y)(x-y)+z(2x^2-10y^2-7xy)+z^2(x+y)+z^3=0,$$

or, $$(x+2y+c_1z)(3x+2y+c_2z)(x-y+c_3z)+z^2(\mathrm{A}x+\mathrm{B}y+\mathrm{C}z)=0$$

whence $3c_1+c_2+3c_3=2, \quad -c_1+8c_3=-7, \quad -2c_1-2c_2+4c_3=-10.$

$$\therefore \quad c_1=1, \quad c_2=2, \quad c_3=-1.$$

Thus the asymptotes are—

$$x+2y+z=0, \quad 3x+2y+2z=0, \quad x-y=z.$$

Notice that the values of c_1, c_2, c_3, satisfy the equation—

$$\frac{2x^2-7xy-10y^2}{(x+2y)(3x+2y)(x-y)}=\frac{c_1}{x+2y}+\frac{c_2}{3x+2y}+\frac{c_3}{x-y}$$

and therefore, in general, the values of c_1, c_2, $c_3 \dots c_n$ are determined by putting $\frac{u_{n-1}}{u_n}$ into partial fractions. *

Ex. 2. The n asymptotes of an n-ic pass through a point O. Show that O is the mean centre of the points where any line through O meets the curve.

* For a detailed account of the different methods for determining the asymptotes of a curve, the reader is referred to Edward's Diff. Calc., Chapter VIII, and Salmon's H. P. Curves, Chap. II, § 52.

Ex. 3. Show that the four asymptotes of the curve

$$xy(x^2-y^2)+(x^2+y^2)=a^2$$

cut the curve again in eight concyclic points.

Ex. 4. Find the equation of a cubic through (a, b) having the lines $x=0$, $y=0$, $y=x$, as asymptotes which cut it in points lying on the line $x+y=a$.

Ex. 5. Prove that the mn intersections of two curves of the mth and nth orders and the mn intersections of their asymptotes lie on a curve of order $(m+n-2)$. (Math. Tripos, 1876.)

193. Asymptotic Curves :

Suppose there are two curves which approach each other continually, so that for the same value of the abscissa (or ordinate) the limit of the difference of the ordinates (or abscissae) is zero, when the common abscissa (or the ordinate) is infinite. These two curves are said to be *asymptotic* to each other.

The two curves $ax^3+bx^2-xy+cx+d=0$
and $ax^2+bx+c-y=0$
are asymptotic.

For, these equations can be written as—

$$y=ax^2+bx+c+\frac{d}{x} \quad \text{and} \quad y=ax^2+bx+c.$$

The difference of the ordinates at any point x is d/x, whose limit is zero, when $x=\infty$.

Ex. 1. Show that the curve $x^2y=x^3+x^2+x+1$ has a hyperbolic asymptote.

The equation can be written as $y=x+1+\dfrac{1}{x}+\dfrac{1}{x^2}$

Hence, the equation of a curve asymptotic to this is the hyperbola

$$y=x+1+\frac{1}{x}, \quad \text{or,} \quad x^2-xy+x+1=0,$$

whose rectilinear asymptotes are $y=x+1$ and $x=0$.

Ex. 2. Show that the curve $y^3-xy-1=0$ has a parabolic asymptote.

The equation can be written as $x=\dfrac{y^3-1}{y}=y^2-\dfrac{1}{y}$.

$\therefore$ The parabola $x=y^2$ is asymptotic to the curve.

Ex. 3. Show that the curve $axy=y^3-a^3$ has a parabolic asymptote,

194. Parabolic Branches:*

Suppose the equation of a curve is thrown into the form

$$(y-mx)^2\phi+z\psi+\ldots=0$$

i.e., the terms of the highest degree in x and y has a double factor. It is evident from the equation that the line at infinity touches the curve at a point in the direction of the line $y=mx$.

The equation may be written as—

$$(y-mx)^2+\frac{zx\psi}{x\phi}+\ldots\ =0$$

If we put a for Lt. $\dfrac{\psi}{x\phi}$, and b for Lt. $\dfrac{\psi}{\phi}$, when x and y become infinite in the ratio $1:m$, the curve ultimately approximates to the parabolic form—

$$(y-mx)^2+axz+bz^2+\ldots=0 \qquad \ldots \quad (1)$$

This parabola is a first approximation to the shape of the curve, but is not in general asymptotic to it, and serves only to suggest the method of closely examining the parabolic branches. In practice, it is found more convenient to adopt a method of successive approximation to obtain the ultimate form of the parabolic branch at an infinite distance. The method will be best illustrated by the following examples:—

Ex. 1. Examine the form at infinity of the curve $x^6+2x^3y-y^3=0$.

There are no asymptotes parallel to the axes. The only infinite branch is, where $y:x$ is large, of the form $x^2=y$, where x^3y varies as x^5, and may be neglected compared with the term x^6 retained in the first approximation.

* For a detailed discussion of curvilinear asymptotes, the reader is referred to Frost's Curve Tracing, Chapters VII and VIII.

The parabola $x^2 = y$ is not a proper asymptote, but serves to suggest the direction of the infinite branches. The proper asymptotes may be obtained by approximation : Thus, from the equation we have—

$$y = x^2\left(1 + \frac{2y}{x^3}\right)^{\frac{1}{3}}$$

$$= x^2\left\{1 + \frac{1}{3}\frac{2y}{x^3} - \frac{4}{9}\frac{y^2}{x^6} + \ldots\ldots\right\}$$

By putting $\frac{y}{x^2} = 1$, the second approximation is $y = x^2\left(1 + \frac{2}{3x}\right)$

The third approximation is $y = x^2\left\{1 + \frac{2}{3x}\left(1 + \frac{2}{3x}\right) - \frac{4}{9x^2}\right\}$

$$= x^2 + \tfrac{2}{3}x.$$

$\therefore$ The proper parabolic asymptote is $y + \frac{1}{9} = (x + \frac{1}{3})^2$.

Ex. 2. Show that the curve $y + x + a = \dfrac{a^3}{3x^2}$

is asymptotic to the Folium of Descartes $x^3 + y^3 = 3axy$.

Ex. 3. Examine the parabolic branch of the curve—

$$x^3 + aby - axy = 0.$$

195. Circular Asymptotes :

When the equation of a curve is given in polar co-ordinates, it may happen that θ being increased indefinitely, the value of r tends to a fixed limiting value, and the curve approaches more and more nearly to the circular form at the same time. The equation takes the form of one in r representing one or more circles. Such a circle is called an *asymptotic circle* of the curve.

Ex. 1. Consider the curve $r = a\theta^2/(\theta^2 - 1)$.

The equation may be written as $r = a/(1 - 1/\theta^2)$

Now, when θ becomes very large, $1/\theta^2$ is very small, and the curve approaches indefinitely near the limiting circle $r = a$.

Ex. 2. Find the asymptotic circles of the curves :

(i) $r = a\dfrac{\theta}{\theta - 1}$ (ii) $r = \dfrac{a}{\theta \sin \theta} + b$ (iii) $r = \dfrac{a\theta}{\theta + \sin \theta}$

Ex. 3. Find the circular asymptote to the curve $r = \dfrac{a\theta + b}{\theta + a}$.

Section II : Tracing of Curves :

196. Curve Tracing in Cartesian Co-ordinates:

If we give any value 'a' to x, the resulting equation in y can be solved and will determine the points in which the line $x=a$ intersects the curve. By giving a set of values to x, we determine a corresponding set of values of y, and thereby obtain a number of points on the curve, sufficient to give a fairly good idea of the shape of the curve.

In particular, we may easily determine the points where the curve cuts the axes of co-ordinates, by putting $x=0$, and $y=0$ respectively in the equation. Again, if y becomes imaginary for any value of x, we guess the existence of an oval, or that the curve is limited in any direction.

The value of $\partial y/\partial x$ at any point gives the direction of the tangent at that point, and where its value becomes zero or infinite, the tangent, and consequently the direction of the curve, is parallel or perpendicular to the axis of x. In some cases the equation of the curve suggests simplifications and taking advantage of these, we can more easily obtain a fairly accurate shape of the curve.

For illustrations the student is referred to Frost's Curve Tracing and to Cramer's Introduction to the Analysis of Curves.

197. Curve Tracing in Homogeneous Co-ordinates:

In tracing curves whose equations are given in homogeneous co-ordinates, we have to determine the points in which it intersects the sides of the fundamental triangle. One advantage in this system is at once evident. Instead of two lines we have three sides and three vertices, any one of which can lie on the curve. Hence the form of the equation gives more information about the curve than in the Cartesian system. On the other hand, there is one disadvantage that homogeneous co-ordinates cannot be

used in the measurement of actual lengths. Consequently, this system can be used in studying the general nature of a curve, but it gives us no information about its metrical properties. The process of tracing curves in homogeneous co-ordinates may be illustrated by a few examples :

Ex. 1. Trace the curve $x^4+x^2y^2-y^2z^2=0$.

Here $x=0$ gives $y^2z^2=0$. Therefore the curve passes through the vertices B and C. The vertex B is a double point, since the co-efficients of y^4 and y^3 are absent. The tangents at this point are—

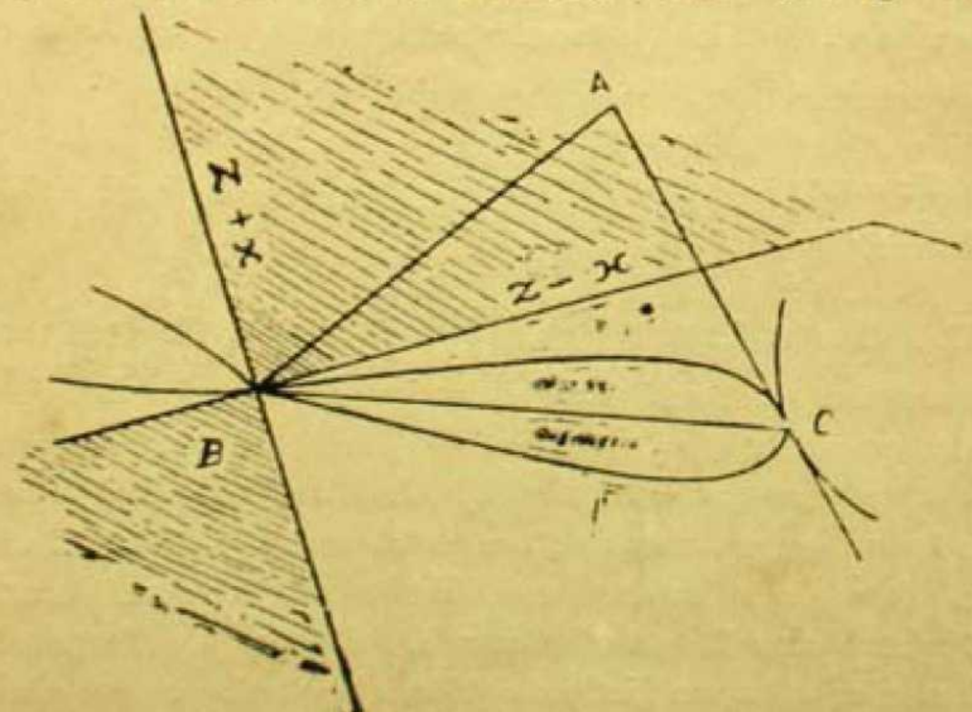
$$x^2-z^2=0, \ \ i.e., \ x\pm z=0.$$

Similarly, there is a double point at C, the tangent at this point being $y=0$.

Since $y=0$ gives $x^4=0$, all the four points on AC are coincident at C; also $z=0$ makes $x^2(x^2+y^2)=0$, and consequently two points on AB are coincident at B and two other are imaginary on the lines $x\pm iy=0$.

To determine the shape of the curve, we place the equation on the analytical triangle.

Consider the vertex C (III). The equation after putting $z=1$



(§ 188) becomes $x^4+x^2y^2-y^2=0$, and the shape at C is given by $x^4-y^2=0$, *i.e.*, by $x^2\pm y=0$.

Hence at C there are two parabolic branches $x^2 \pm y = 0$ forming a *tacnode*,* and the tangent at C has a four-pointic contact.

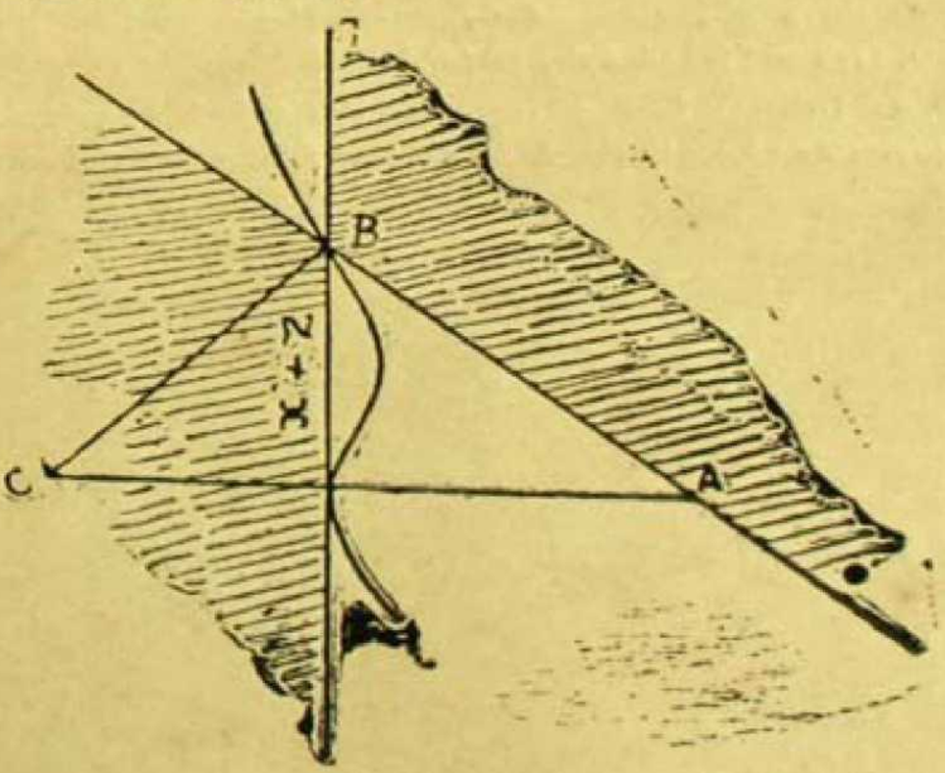
Similarly, to determine the shape at B, we put $y = 1$ and the equation becomes $x^4 + x^2 - z^2 = 0$. The tangents at B are $x \pm z = 0$ and the shape is given by $z^2 = x^2(1 + x^2)$ or $z = x + \frac{1}{2}x^3 + \ldots$

$\therefore$ $z - x$ is an inflexional tangent. Similarly, $z + x$ is also an inflexional tangent.

Hence the point B is a *biflecnode* † on the curve. Again, since the equation may be written as $x^4 = y^2(z + x)(z - x)$, it follows that $z + x$ and $z - x$ must always have the same sign. Thus the curve cannot lie in the shaded parts of the plane, as shown in the adjoining diagram.

Ex. 2. Draw the curve $x^3 - x^2z - y^2z = 0$.

The curve evidently passes through the vertices B and C. The tangent at B is $z = 0$, *i.e.*, the side AB. The point C is a conjugate point on the curve, the tangents being $x \pm iy = 0$.



The curve meets $x = 0$, where $y^2z = 0$, *i.e.*, $y^2 = 0$ and $z = 0$. When $y = 0$, we have $x^2(x - z) = 0$.

* A tacnode is formed by the union of two nodes.

† See § 58.

$\therefore$ The side AC meets the curve at C and at the point $y=0$, $z=x$. When $z=0$, $x^3=0$, or the side AB is an inflexional tangent at B.

The first approximation to the shape at B is given by $x^3-y^2z=0$, *i.e.*, $x^3-z=0$, which is a cubical parabola. The point B is an inflexion.

Again, the equation can be written as $\quad x^3=z(x^2+y^2)$

$\therefore$ z and x must always be of the same sign.

Also $x^2(x-z)=y^2z$. $\therefore$ $x-z$ and z must be of the same sign. Hence the curve cannot lie in the shaded parts of the plane, and its form is shown in the figure. The line $z-x=0$ is a tangent at the point $y=0$, $z-x=0$. The asymptote of the cubic is imaginary.

Ex. 3. Trace the following curves :

(*i*) $y(x-az)^2=x(y-4az)^2$ $\qquad$ (*ii*) $x^5-a^3x^2z^3+y^5=0$

(*iii*) $azx(y-x)^2-y^4=0$.

Ex. 4. Trace the curve $xy^2=4a^2\ (2a-x)$.

[This curve was discussed by Prof. Maria Gaetana Agnesi of Bologna, 1748 and is called the *Witch*.]

Ex. 5. Indicate how a curve can be traced by points in polar co-ordinates, and trace the curve $r=a+b\cos\theta$ (Limaçon of Pascal).

CHAPTER X

Rational Transformations

198. In Chapter I, we have discussed particular methods of transformations, such as Reciprocation, Inversion, Projection, etc., and thereby from known properties of one curve deduced those of another derived from it by any of these processes. In the present chapter, however, we shall study the general principles of these methods, which consists chiefly in instituting a relation between any two points P and P′ in the same plane, or in different planes lying in a common space. The case of different planes properly belongs to space-geometry, and consequently without any reference to space, we shall regard the planes as superimposed one upon the other, so as forming a single plane. Thus we shall have to consider two figures consisting of points, lines, etc., in the same plane, instead of two figures in different planes.

199. Rational and Birational Transformations:

If (x, y, z) and (x', y', z') be the homogeneous co-ordinates of two points P and P′ in two planes (or same plane), α and β respectively, then the transformation, expressed by the equations

$$x' : y' : z' = f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z) \quad \dots \quad (1)$$

where f_1, f_2, f_3 are known functions of x, y, z, each of order n (say) without a common factor, indicates that to any system of values of x, y, z, there corresponds a single system of values of x', y', z'; but to a given system of values of x', y', z' there will not, in general, correspond a single system but a finite number $(D \geq 1)$ of values of x, y, z.

Thus, when f_1, f_2, f_3 are rational, there is an algebraic relation between (x, y, z) and (x', y', z') expressed by the equations (1), which is called a *Rational Transformation*.

If, however, $D=1$, *i.e.*, if to a given system of values of x', y', z' there corresponds a single system of values of x, y, z, expressed by—

$$x : y : z = F_1(x', y', z') : F_2(x', y', z') : F_3(x', y', z') \quad \dots \quad (2)$$

F_1, F_2, F_3 must also be rational and of order n, and when such mutual expression is possible, the relation is called a *Birational Transformation*, or Cremona Transformation, after the name of Cremona who first studied it.*

200. Linear Transformations:

DEFINITION: Any transformation by which two figures are so related that any point and line of one correspond to one, and only one point and line respectively of the other, and conversely, is called a *linear homographic* transformation.

Since the correspondence must be (1, 1), the required expressions cannot contain any radicals.

Thus, $\quad x' : y' : z' = f_1 : f_2 : f_3$

where f_1, f_2, f_3 are algebraic functions and polynomials in x, y, z, having no common factor.

Since, to any straight line $lx'+my'+nz'=0$ corresponds

$$lf_1 + mf_2 + nf_3 = 0,$$

which must be a right line for all values of l, m, n, the functions f_1, f_2, f_3 must be linear in x, y, z.

Thus $x' : y' : z' = ax+by+cz : a'x+b'y+c'z : a''x+b''y+c''z$, whence x, y, z can be expressed linearly in terms of x', y', z'.

Hence, assuming proper triangles of reference and the ratios of the implicit constants, we may write, without loss of generality,

$$x' : y' : z' = x : y : z.$$

* Cremona—Bologna Mem. (2) Vol. 2 (1863), p. 621, and Vol. 5 (1864), p. 3, or, Giorn. di mat., Vol. 1 (1863), p. 305, Vol. 3 (1865), pp. 263 and 363.

The above transformation contains eight independent constants, and consequently, any four points (or lines) of one figure can be made to correspond to four points (or lines) in the other. Therefore this transformation leaves the cross-ratio of any four elements unaltered. so also the order or class of a curve remains unchanged.

The method of projection explained in § 12 is a particular case of linear homographic transformation, which involves only five constants, the vertex and axis of projection reducing the constants by three.

201. Collineation:

The linear transformation admits of a double interpretation. It may be regarded as a transformation of co-ordinates, or as a relation between the points of two different (or superimposed) planes. Let us imagine that the planes are infinitely near one another (or superimposed) and suppose that the points are referred to the same triangle of reference.

If then $P(x, y, z)$ and $P'(x', y', z')$ represent points in the two planes respectively, the linear equations—

$$\left.\begin{aligned} x' &= ax + by + cz \\ y' &= a'x + b'y + c'z \\ z' &= a''x + b''y + c''z \end{aligned}\right\} \quad \cdots \quad (1)$$

establish between the points of the two planes a (1, 1) correspondence, which is called "*linear affinity*" or *collineation*,* such that to each point P of one plane corresponds a point P′ (point-image of P) in the other. This relation, however, is not reciprocal, *i.e.*, to the point P′ does not, in general, correspond the same point P, but the corresponding point in the first plane is obtained by

* This was first discussed by Möbius—Barycentric Calculus (1827), p. 266—and afterwards by Magnus—Aufgaben and Lehrsätze aus der analytischen Geometrie, Berlin (1833).

solving the equations (1), provided the determinant $\triangle$ of the co-efficients does not vanish.

Thus,
$$\triangle x = Ax' + By' + Cz'$$
$$\triangle y = A'x' + B'y' + C'z'$$
$$\triangle z = A''x' + B''y' + C''z'$$

where A, A′, A″, etc., denote the minors of a, a' a'', etc. These formulæ are evidently the same as for the transformation of co-ordinates, where the variables are the co-ordinates of the same point referred to two different triangles of reference, while in the present case, they are the co-ordinates of two different points referred to the same triangle.

Thus it will be seen that if P describes a curve in one plane, P′ describes the corresponding curve of the same order in the other plane, and in particular, a straight line corresponds to a straight line, a range of points or pencil of lines corresponds to a projective range of points or pencil of lines respectively.

202. Collineation treated geometrically:

The geometric determination of collineation is contained in the following theorem:

If to four points of one plane, no three of which are collinear, there correspond in the other four points, no three of which are collinear, the linear relation, *i.e.*, collineation between the points of the two planes is completely determined.

For, if we are given four pairs of corresponding points, the equations (1) of the preceding article are uniquely determined.

The following geometrical construction is useful and interesting:

To the line joining any two points of one plane Σ_1 corresponds, in each case, the line joining the two corresponding points of the second plane Σ_2, and to the

point of intersection of two lines in Σ_1 corresponds the point of intersection of the corresponding lines of Σ_2. Now, the four points of Σ_1 determine six lines, which again determine three new points, namely, the diagonal points of the complete quadrilateral. To these then correspond the three points of the plane Σ_2 obtained by a similar construction. If now the lines joining the three points of Σ_1 are produced to meet the six sides of the complete quadrilateral, we obtain new points of Σ_1 whose correspondents in Σ_2 are constructed exactly in a similar manner. Thus, the entire plane will be covered over by a net-work of lines, and by the continuous crossing of the meshes of these nets, we shall obtain a point indefinitely near to a point of Σ_1. If the corresponding nets are constructed in Σ_2, the respective collineation of the individual points of the two planes is established by the nets, and consequently the collineation is completely determined.*

203. The Dualistic Transformation : †

We have thus far considered only linear transformations in which a point corresponds to a point and a line to a line; but there are transformations where a point corresponds to a curve, for example, in Reciprocation a point corresponds to a line and *vice versa*. Such a transformation is called "*Skew Reciprocation*" or "Linear Dualistic Transformation." Reciprocation as described before is a special case of this more general linear dualistic transformation and differs from it only by a linear transformation.

* Möbius—Bar. Cal., p. 273. For analytical treatment, the student is referred to Scott—Modern Analytical Geometry, §§ 223-26.

† For a detailed account of the theory, see Salmon's H. P. Curves, §§ 332-42, or Scott—*ibid*, §§ 253-56.

Let (x, y, z) be a system of point co-ordinates and (ξ, η, ζ) a system of line co-ordinates in the same or in different planes. Then, a point in the first system corresponds to a line in the second, if the co-ordinates of the point are proportional to the co-ordinates of the line, *i.e.*, $x : y : z = \xi : \eta : \zeta$ and consequently, to any line $lx+my+nz=0$ corresponds the point $l\xi+m\eta+n\zeta=0$.

In the general dualistic transformation, however, the co-ordinates of a line are functions of the co-ordinates of the corresponding point, and the transformation is linear when those functions are linear.

Thus,
$$\left.\begin{aligned} \xi &= a_1x+b_1y+c_1z \\ \eta &= a_2x+b_2y+c_2z \\ \zeta &= a_3x+b_3y+c_3z \end{aligned}\right\} \quad \ldots \quad (1)$$

where to a point (x, y, z) there corresponds the line (ξ, η, ζ) in the same or different planes. If, however, we put—

$$x' : y' : z' = a_1x+b_1y+c_1z : a_2x+b_2y+c_2z : a_3x+b_3y+c_3z$$

i.e., if a point (x', y', z') is obtained corresponding to the point (x, y, z) by a linear transformation, there is a correspondence between the point (x', y', z') and the line (ξ, η, ζ), and we have, as stated above, the following relations:

$$x' : y' : z' = \xi : \eta : \zeta.$$

This shows that the systems (x', y', z') and (ξ, η, ζ) are reciprocal with respect to the auxiliary conic

$$x^2+y^2+z^2=0.$$

Thus, *the linear dualistic transformation differs from the interchange of point and line co-ordinates only by a collineation.*

204. Dual System :

If we solve the equations (1) of the preceding article for x, y, z, we obtain the following relations:

$$\left.\begin{aligned}\Delta x&=A_1\xi+A_2\eta+A_3\zeta\\ \Delta y&=B_1\xi+B_2\eta+B_3\zeta\\ \Delta z&=C_1\xi+C_2\eta+C_3\zeta\end{aligned}\right\} \quad \ldots \quad (2)$$

where A_1, B_1, etc., are the minors of a_1, b_1, etc., in the determinant Δ of the co-efficients in (1).

The system (2) is said to be "*dual*" of the system (1).

Now consider the point (x', y', z') in the first system. Its corresponding line in the second system is then :—

$$x'(a_1x+b_1y+c_1z)+y'(a_2x+b_2y+c_2z)$$
$$+z'(a_3x+b_3y+c_3z)=0$$

or, $x(a_1x'+a_2y'+a_3z')+y(b_1x'+b_2y'+b_3z')$

$$+z(c_1x'+c_2y'+c_3z')=0 \quad \ldots \quad (3)$$

The equation (3) expresses the relation between any point (x', y', z') of the first system and any point (x, y, z) on a corresponding line of the dual system. If now (x, y, z) is considered fixed and (x', y', z') variable, we have for the line of the first system, corresponding to any point of the second,

$$x(a_1x'+b_1y'+c_1z')+y(a_2x'+b_2y'+c_2z')$$
$$+z(a_3x'+b_3y'+c_3z')=0 \quad \ldots \quad (4$$

The lines (3) and (4) do not, in general, coincide; hence, in the general dualistic transformation, every point has a different corresponding line, according as the point is regarded as belonging to the first or to the second system.

The conditions that the lines (3) and (4) should coincide give three values of (x', y', z'). Hence there are three points in the plane associated with their corresponding lines in a definite way, regardless of the system to which they belong.

One of these points, however, is given by—

$$x' : y' : z' = c_2 - b_3 \;:\; a_3 - c_1 \;:\; b_1 - a_2$$

and the other two are real or imaginary.

The two lines (3) and (4) will coincide for all points of the plane, if for all values of x', y', z', we have

$$a_1x' + b_1y' + c_1z' : a_2x' + b_2y' + c_2z' : a_3x' + b_3y' + c_3z'$$

$$= a_1x' + a_2y' + a_3z' : b_1x' + b_2y' + b_3z' : c_1x' + c_2y' + c_3z'$$

which requires $c_2 = b_3$, $c_1 = a_3$ and $a_2 = b_1$.

Hence, the transformation formulæ reduce to the forms—

$$\xi = ax + hy + gz$$

$$\eta = hx + by + fz$$

$$\zeta = gx + fy + cz$$

This shows that the point and the line are associated with each other as pole and polar with regard to the general conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Thus it is seen that in case of reciprocals with regard to a conic, the same line corresponds to a point, whether that point be considered as belonging to the first or the second system.

205. Pole and Polar Conics:

The case of a point lying on its corresponding line is interesting and deserves consideration.

Since a point (x, y, z) lies on its corresponding line $\xi x + \eta y + \zeta z = 0$, the locus of such points is obviously—

$$(a_1x + b_1y + c_1z)x + (a_2x + b_2y + c_2z)y + (a_3x + b_3y + c_3z)z = 0$$

i.e., $$a_1x^2 + b_2y^2 + c_3z^2 + (b_3 + c_2)yz + (a_3 + c_1)zx + (a_2 + b_1)xy = 0 \quad \ldots \quad (1)$$

and this is the same conic, whether the point be

considered as belonging to the first or to the second system, and is called the "*Pole Conic*."

On the other hand, the envelope of lines which pass through their corresponding points is a conic called the *Polar Conic*.

The co-ordinates of the point are expressed in terms of the co-ordinates of their corresponding lines by the equations (2) of § 204. Therefore the required envelope is

$$(A_1\xi+A_2\eta+A_3\zeta)\xi+(B_1\xi+B_2\eta+B_3\zeta)\eta$$
$$+(C_1\xi+C_2\eta+C_3\zeta)\zeta=0$$

i.e., $$A_1\xi^2+B_2\eta^2+C_3\zeta^2+(B_3+C_2)\eta\zeta$$
$$+(A_3+C_1)\xi\zeta+(A_2+B_1)\xi\eta=0 \quad \ldots \quad (2)$$

where A_1, B_1, C_1, etc., have the significance as in § 204.

Conversely, the same pole and polar conics will be obtained, if the points of the second system correspond to the lines of the first system.

The pole and the polar conics have double contact, the intersection of the common tangents being the point (b_3-c_2), (c_1-a_3), (a_2-b_1). The chord of contact is found to be the line (B_3-C_2), (C_1-A_3), (A_2-B_1).

It will be seen that the pole and polar conics are identical, if $b_1=a_2$, $b_3=c_2$ and $c_1=a_3$.*

206. Quadric Inversion:

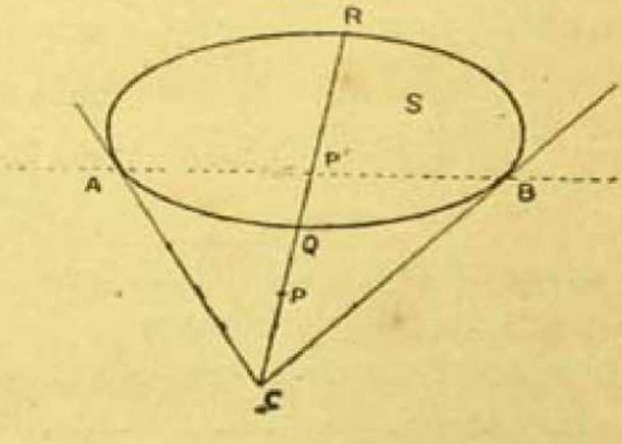
The process of circular inversion has already been described in § 15; but in this section will be described a more general process in which a point corresponds to a point, while a line, in general, corresponds to a conic. This transformation can easily be effected by a geometrical construction and was given by Dr. Hirst.† In this process a fixed point is taken as origin and a fixed

* Scott—*loc. cit.*, § 256.

† Hirst—"*On the Quadric Inversion of Plane Curves*," Proc. of the R. Soc. of London, Vol. 14 (1865), pp. 91-106.

fundamental conic as "base." Points collinear with the origin and conjugate with respect to the base are said to be *inverse*. If the base is a circle and the origin its centre, the points are ordinary inverse points with regard to the circle. It is, in fact, the circular inversion generalised and is called *Quadric Inversion*.

Let C be the fixed origin and S the base-conic. Through C draw a transversal cutting the base in the points Q, R. Then, if P, P′ are points on the transversal, such that (PP′, QR) is harmonic, then P and P′ are inverse points.



Thus, to determine the inverse of a point P, we have to find the point P′, where CP intersects the polar line of P with regard to S. It follows hence that to any position of P corresponds a single definite position of P′, and *vice versa*.

If P traces out a locus Σ, P′ will trace out a locus Σ', and Σ' is said to be derived from Σ by quadric inversion.

207. Analytical Treatment:

Let CA and CB be the tangents to the base, and choose ABC as the triangle of reference. Then the equation of the base-conic may be written as—

$$xy=z^2 \qquad \dots \quad (1)$$

Let (x, y, z) and (x', y', z') be the co-ordinates of P and P′ respectively. Now, the polar line of P′ is—

$$xy'+yx'-2zz'=0 \qquad \dots \quad (2)$$

and the line CP′ is

$$xy'-x'y=0 \qquad \dots \quad (3)$$

whence $x : y : z = x' : y' : x'y'/z'$

$$= x'z' : y'z' : x'y' = \frac{1}{y'} : \frac{1}{x'} : \frac{1}{z'}.* \qquad (4)$$

If $f(x, y, z) = 0$ is the locus of P, the locus of P′ is given by the equation—

$$f\left(\frac{1}{y'}, \frac{1}{x'}, \frac{1}{z'}\right) = 0.$$

Applying the linear transformation $x' : y' : z' = y' : x' : z'$, *i.e.*, interchanging the vertices A and B of the triangle, we may express the result (4) in a more symmetrical form, and the locus of P′ is now given by—

$$f\left(\frac{1}{x'}, \frac{1}{y'}, \frac{1}{z'}\right) = 0.$$

The formulæ of transformation can, however, be written under the form of bilinear relations—

$$xx' = 1, \quad yy' = 1, \quad zz' = 1.$$

208. Quadric Inversion as Rational Transformation:

Let the formulæ of transformation in § 199 be put into the form—

$$x' : y' : z' = f_1 : f_2 : f_3$$

where f_1, f_2, f_3 are rational functions of the second degree in x, y, z.

To the lines $x' = 0$, $y' = 0$, $z' = 0$ will then correspond the three conics $f_1 = 0$, $f_2 = 0$, $f_3 = 0$; and in general, to a curve of order n corresponds one of order $2n$, obtained by putting f_1, f_2, f_3 respectively for x, y, z in the equation of the n-ic.

The simplest case presents itself in the form—

$$x' : y' : z' = x^2 : y^2 : z^2.$$

* These formulæ are deduced on the supposition that the base is a proper conic and the points A, B, C are distinct. Modifications are necessary when the base is a degenerate conic, and two or more of A, B, C are coincident.

To the line $lx+my+nz=0$ corresponds the conic $l\sqrt{x}+m\sqrt{y}+n\sqrt{z}=0$ inscribed in the triangle xyz. Similarly, to a conic there corresponds a curve of the fourth order, and so on.

It is to be noticed, however, that this transformation is not birational in general. For, although x', y', z' are expressed rationally in terms of (x, y, z), the latter are given in terms of x', y', z' by the equations—

$$\frac{f_1}{x'}=\frac{f_2}{y'}=\frac{f_3}{z'}$$

which are not rational and represent conics having four common points; and consequently, corresponding to any position of (x', y', z'), there are *four* positions of (x, y, z).

But if f_1, f_2, f_3, have one common point, since it is independent of the position of (x', y', z'), it may be ignored, and to any position of (x', y', z') there will correspond only *three* points (x, y, z). Similarly, if f_1, f_2, f_3 have two points common, to any position of (x', y', z') will correspond only *two* positions of (x, y, z). Finally, if f_1, f_2, f_3 have three common points, the conics have, besides the three common points, only *one* other common point, and to any position of (x', y', z') there corresponds only a single position of (x, y, z), and *vice versa*, and the transformation is birational.

Since it is perfectly legitimate to take three conics of the form $lf_1+mf_2+nf_3$ instead of f_1, f_2, f_3, the three line-pairs joining each of the three common points to the other two may be taken for f_1, f_2, f_3, and the formulæ become :

$$x : y : z=y'z' : z'x' : x'y'$$

and $$x' : y' : z'=yz : zx : xy.$$

Hence, the quadric inversion is only a particular case of the general birational transformation.

Other special cases may arise from the coincidence of two or more of the common points.

Thus, when two points coincide, we may take the common tangent as the side $y=0$ and the point (z, x) as the third common point. The equations of f_1, f_2, f_3 will be of the form $$ax^2+2fyz+2hxy=0.$$

Taking x^2, yz, xy as the three conics, the formulæ become
$$x' : y' : z' = xy : x^2 : yz$$
and
$$x : y : z = x'y' : x'^2 : y'z'$$

Similarly, when the three points coincide, the equations of f_1, f_2, f_3 will be of the form—
$$by^2+2hxy+2f(yz-mx^2)=0$$

Hence, taking y^2, xy, $yz-mx^2$ for f_1, f_2, f_3 respectively, the formulæ become:
$$x' : y' : z' = xy : y^2 : yz-mx^2$$
and
$$x : y : z = x'y' : y'^2 : y'z'-mx'^2$$

209. Inverse of Special Points.

It has been stated in § 206 that the inverse of a point P is a single definite point P′. But there are exceptional positions of P for which the inverse point P′ is not in general determinate. The inverse P′ is indeterminate, if

(*i*) P is at C, P′ is any point on AB,
(*ii*) P is at B, P′ is any point on BC,
(*iii*) P is at A, P′ is any point on CA,
(*iv*) P is any point on AB, P′ is at C,
(*v*) P is any point on BC, P′ is at B,
(*vi*) P is any point on CA, P′ is at A.

Hence it appears that if P is at any vertex or on any side of ABC, the ordinary laws of correspondence do not apply.

210. The Inverse of a Straight Line:

The inverse of the straight line $lx+my+nz=0$ ... (1) is the locus defined by the equation (§ 207)

$$l/y+m/x+n/z=0 \qquad \ldots \quad (2)$$

which evidently represents a conic circumscribing the triangle ABC.

The following special cases are to be noted:

(i) If the line (1) passes through C, $n=0$ and it is its own inverse.

(ii) If the line passes through A, $l=0$ and the inverse is the line $mz+nx=0$, which passes through B.

(iii) If the line passes through B, its equation is $lx+nz=0$, and the inverse is the line $lz+ny=0$ through A.

Thus, it is seen that the inverse of a right line is, in general, a conic through A, B, C; but in special cases it is a right line.

211. Proper Inverse:

Let O be the pole of a line meeting the base-conic in Q and R. Then the inverse of the line is the conic ABCOQR. But if the line passes through C, then the pole O lies on AB, and the conic has three points on AB, *i.e.*, the conic consists of AB and the given line CQR. The line AB presents here as a part of the inverse simply because the inverse of C is indeterminate, being any point on AB.

When the line passes through A or B and meets the base-conic in another point K, the inverse is a degenerate conic composed of CA, BK or CB, AK; for the pole O is now on CA or CB.

Hence the points C, A, O on the conic are accounted for by the line CA or CB and the remaining points B or A and K give the other line.

Similarly, if a curve passes through A (or B), the line CA (or CB) presents itself as part of the inverse. These factors, however, occurring in the inverse are not regarded

as forming the proper inverse and are rejected. The remaining factor gives the proper inverse.

Ex. Consider the conic $fyz+gzx+hxy=0$

The inverse of this, by the formulæ of § 208, is—

$$fx'^2y'z'+gx'y'^2z'+hx'y'z'^2=0$$

i.e., $$x'y'z'(fx'+gy'+hz')=0$$

i.e., the sides AB, BC, CA and another line. Hence, rejecting the factor $x'y'z'$, the proper inverse is the line $fx+gy+hz=0$.

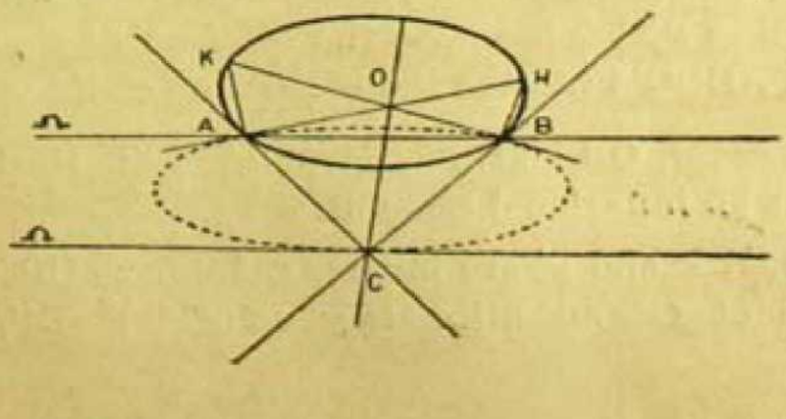
212. The Inverse of the Line at Infinity :

The equation of the line at infinity being $ax+by+cz=0$, its inverse, by the formulæ of § 207, is the conic—

$$azx+byz+cxy=0 \qquad \dots \quad (1)$$

which is evidently a conic circumscribing the fundamental triangle.

The pole O now becomes the centre of the base-conic, the points Q, R (§ 211) are the points at infinity on the same. The polar of the point Ω at infinity on AB passes through C and the line $C\Omega$ is parallel to AB. Therefore the inverse of Ω is consecutive to C on the line $C\Omega$, or in other words, the tangent to the inverse (1) at C is parallel to AB. Thus OC is the diameter conjugate to AB and the tangent at C is parallel to AB. It may be noticed further that if the line drawn through A parallel to CB meets the base-conic in K, BK is the tangent at B.



Similarly, the tangent at A may be constructed. The inverse to the line at infinity is represented in the figure by the dotted line.

213. Inversion of Special Points on a Curve :

Let Σ be the curve and Σ' its inverse. The following special points are to be noticed :

If Σ meets AB in P, Σ' touches CP at C. For, the inverse of P is on CP by definition, and as the polar of P passes through C, the inverse is indefinitely near to C on CP. Hence CP is the tangent at C. Similarly, if Σ cuts AB at n points, other than A, B, the inverse has an *n-ple* point at C. For instance, if the curve cuts AB at two points P_1 and P_2, there is a node at C with CP_1 and CP_2 as tangents. When P_1 and P_2 become consecutive points on Σ, *i.e.*, AB is a tangent to Σ at P_1, the two tangents CP_1, CP_2 coincide, and there is a cusp on Σ' at C, with CP_1 as the cuspidal tangent.

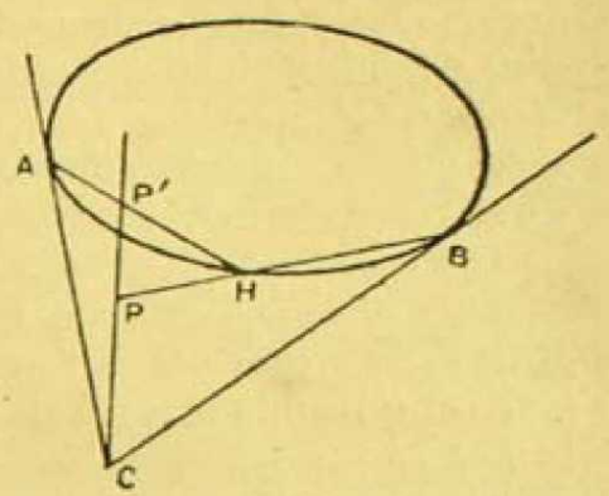
To the tangent to Σ at P corresponds the conic osculating Σ' at C and passing through A, and B.

For, if Q is a point on Σ consecutive to P, the inverse Q$'$ is consecutive to C on Σ' and the limiting position of CP$'$Q$'$ (*i.e.*, CP) is the tangent to Σ' at C. Now the inverse of PQ is a conic through A, C, B, Q$'$, touching Σ' at C (Q$'$ being the point on Σ' inverse to Q). If then Q approaches P, PQ becomes the tangent to Σ at P, and the conic ABCQ$'$ becomes the osculating conic of Σ' at C.

If, however, CP be the tangent to Σ at P, the inverse has CP as an inflexional tangent.

Again, if P and P$'$ are inverse points, as CP gradually turns about C and ultimately coincides with CA, P

approaches and ultimately coincides with A. Hence AP′ becomes the tangent to the inverse of the locus of P. But AP′ and BP intersect at a point H on the curve. If AP is the line $mz+nx=0$, its inverse is the line AP′ defined by $my+nz=0$, and they intersect at a point H on the conic whose equation is $xy=z^2$. (§ 210.) Thus,



if Σ meets CA at P, the inverse Σ' touches at A the line AH, *i.e.*, the line through A corresponding to BP. Similarly, if Σ meets CB. Hence, to every intersection of Σ with CA (or CB) there corresponds a branch of Σ' through A (or B), and conversely.

Thus, if an *n-ic* Σ cuts CA (or CB) in n points, there is an *n-ple* point on Σ' at A (or B). When Σ touches CA (or CB), the two tangents to Σ' at A (or B) coincide, and consequently, A (or B) is a cusp on Σ'.

214. Effects of Inversion on Singularities:

From what has been said above, it follows that, in general, an ordinary point inverts into an ordinary point. But if three consecutive points at the given point and the three fundamental points A, B, C lie on a conic, their inverses are collinear on the inverse curve, and there is, therefore, an inflexion on the inverse. Thus the inverse of an ordinary point may be either an ordinary point or an inflexion. Similarly, an inflexion is inverted into an ordinary point, unless the inflexional tangent passes through any of the fundamental points.

Again, a double point, in general, inverts into a double point of the same nature ; and consecutive double points invert into consecutive double points, but

the appearance may be slightly altered. Thus a tacnode inverts into a tacnode, an oscnode inverts into an oscnode, but if the three nodes are initially collinear, the oscnode on inversion loses this property, unless the tangent passes through a fundamental point. Similarly, a curved oscnode may be straight on inversion.

In the case of a bitangent, the inverse becomes a conic having double contact with the inverse, unless the bitangent passes through a fundamental point, and then it inverts into a bitangent. Conversely, a bitangent may be gained on inversion.

Thus it follows that as regards points and lines, not belonging to the fundamental triangle, the point singularities of a curve and its inverse are the same, but line singularities are changed. Hence inversion can conveniently be used for analysing singularities on curves.

215. Effects of Inversion on a Curve: *

Let the curve Σ be an *n-ic* having a *q-ple* point at A, an *r-ple* point at B and an *s-ple* point at C. Then, Σ meets AB, BC, CA respectively at $n-q-r$, $n-r-s$, and $n-q-s$ other points. The inverse Σ' has therefore an $(n-q-r)$*-ple*, $(n-r-s)$*-ple* and $(n-q-s)$*-ple* point respectively at C, A, B.

Again, the q intersections of the tangents at A to Σ with BC are points on Σ'. Similarly, Σ' meets CA and AB in r and s points respectively other than A, B, C.

Since Σ' meets AB in $\{(n-r-s)+(n-q-s)+s\}$, *i.e.*, $(2n-q-r-s)$ points, Σ' is of order $(2n-q-r-s)$.

Thus, the inverse of an *n-ic* Σ with a *q-ple* point at A, an *r-ple* point at B and an *s-ple* point at C is a curve Σ', of order $(2n-q-r-s)$, with an $(n-r-s)$*-ple* point at A,

* Effects of inversion on higher singular points will be fully discussed in Chap. XIII.

an $(n-q-s)$-*ple* point at B and an $(n-q-r)$-*ple* point at C.

Putting $\quad n'=2n-q-r-s, \quad q'=n-r-s,$

$$r'=n-q-s, \qquad s'=n-q-r,$$

we may establish a reciprocal relation between Σ and Σ'.

Thus, $\quad n=2n'-q'-r'-s' \qquad q=n'-r'-s'$

$$r=n'-q'-s' \qquad s=n'-q'-r'$$

We shall now show that, in general, the deficiencies of the two curves Σ and Σ' are the same.

Since a q-*ple* point is equivalent to $\frac{1}{2}q(q-1)$ nodes the deficiency p of the first curve Σ is given by—

$$p=\tfrac{1}{2}\{(n-1)(n-2)-q(q-1)-r(r-1)-s(s-1)\}$$

and

$$\begin{aligned} p' &= \tfrac{1}{2}\{(n'-1)(n'-2)-q'(q'-1)-r'(r'-1)-s'(s'-1)\} \\ &= \tfrac{1}{2}\{(2n-q-r-s-1)(2n-q-r-s-2) \\ &\quad -(n-r-s)(n-r-s-1)-(n-q-s)(n-q-s-1) \\ &\quad -(n-q-r)(n-q-r-1)\} \\ &= \tfrac{1}{2}\{(n-1)(n-2)-q(q-1)-r(r-1)-s(s-1)\} \\ &= p. \end{aligned}$$

i.e., the deficiency of a curve is unaltered by quadric transformation.

216. Application of Quadric Inversion:

The process of quadric inversion affords a very convenient method of investigating the properties of one curve from known properties of another. The following examples will illustrate the method.

Ex. 1. Consider a conic cutting the sides of the fundamental triangle in three pairs of points.

Let $$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \dots \quad (1)$$

be the equation of the conic cutting the sides BC, CA and AB in the pairs of points A_1, A_2 ; B_1, B_2 and C_1, C_2 respectively.

The inverse of (1) is the quartic curve—

$$a/x^2+b/y^2+c/z^2+2f/yz+2g/zx+2h/xy=0.$$

The points A, B, C are evidently nodes (§ 213) on the curve with the lines AA_1, AA_2 ; BB_1, BB_2 and CC_1, CC_2 as nodal tangents.

But these lines all touch one and the same conic, and they are inverted into themselves. Hence we have the theorem :—

The nodal tangents of a trinodal quartic touch one and the same conic.

Again, the pairs of tangents drawn from A, B, C to the conic are also inverted into themselves, and their inverses are tangents to the trinodal quartic. Hence, *the six tangents drawn from the three nodes to a trinodal quartic touch one and the same conic.*

Finally, the four bitangents of a trinodal quartic are obtained by the same process from the fact that through three given points, there can be drawn four conics having double contact with a given one, while the inflexional tangents are obtained from the fact that through three given points can be drawn six conics, having three-pointic contact with a given conic.

Ex. 2. Show that the three cuspidal tangents of a tricuspidal quartic are concurrent.

If in *Ex.* 1, the pairs of points A_1, A_2 ; B_1, B_2 ; C_1, C_2 are coincident, then the lines joining the vertices to the points of contact of the inscribed conic with the opposite sides are concurrent. The inverse of the conic is evidently a tricuspidal quartic, having the cusps at the vertices, and the joining lines are inverted into themselves, which are again the cuspidal tangents, whence the truth of the theorem follows.

Ex. 3. Through any point can be drawn two lines touching a trinodal quartic and passing through its nodes.

This follows immediately from the fact that from any point only two tangents can be drawn to any conic. On inversion the conic becomes the trinodal quartic and the two tangents invert into two conics through the nodes touching the quartic, and these evidently pass through the inverse of the given point.

217. Circular Inversion :*

A particular case of quadric inversion is the transformation by reciprocal radii the principles of which have been explained in § 15. If we take $k=1$, the relations between the rectangular co-ordinates of P and P′ are—

$$x'=\frac{x}{x^2+y^2}, \qquad y'=\frac{y}{x^2+y^2},$$

and $$x=\frac{x'}{x'^2+y'^2}, \quad y=\frac{y'}{x'^2+y'^2}$$

whence $$x'+iy'=\frac{1}{x-iy}, \quad x'-iy'=\frac{1}{x+iy}$$

Writing $$X : Y : Z = x-iy : x+iy : 1 ;$$

and $$X' : Y' : Z' = x'+iy' : x'-iy' : 1$$

we obtain the relations $X' : Y' : Z' = YZ : ZX : XY$, or in other words, the transformation is a quadric inversion.

The geometrical significance of these transformations will be best understood from the figure of § 206, if we consider that the points A and B are circular points at infinity, so that the base-conic S now becomes a circle with centre C, and P, P′ are inverse points with respect to the circle. In fact, we have taken the circular lines through the origin and the line at infinity as the sides of the triangle of reference. Hence, circular inversion is a particular case of quadric transformation, and quadric transformation is a generalisation by projection of the process of inversion.

We may deduce a number of theorems from the results established in the preceding articles. Thus, the inverse of a circle is a circle, that of a straight line is a circle through C, and so on.

* Moutard—*Sur la transformation par rayons vecteurs reciproques*—Nouv. Ann. *t.* 3(2), (1864), pp. 306-309.

The inverse of a conic, in general, is a trinodal quartic, the nodes being the origin (C) and the circular points at infinity. If the origin be the focus of the conic, the inverse is a *limaçon* ; if the origin be on the conic, the inverse is a nodal *circular* cubic, the origin being the node.

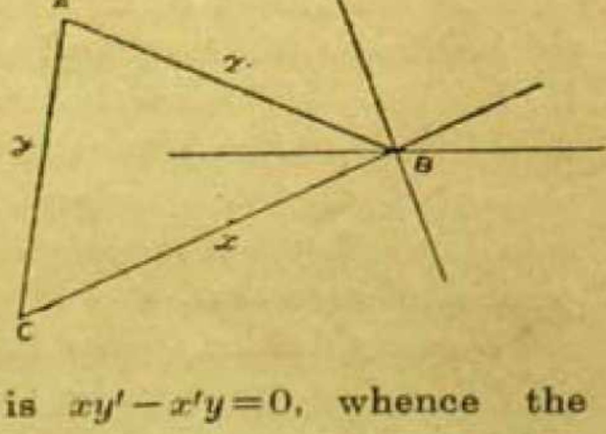
An osculating circle to a curve will invert into an osculating circle of the inverse, but when the circle passes through the origin, the inverse is an inflexional tangent.

218. Special Quadric Transformations:

In § 206 we have discussed the general case of quadric transformation ; but special cases arising from special positions of the points A, B, C or the nature of the *base* must be considered for a systematic treatment of the subject.

Case I: One special case presents itself when the points A and B coincide (§ 208).

In this case, any line through C, the line CA, and the polar of C are taken as the sides of the triangle of reference. The base-conic is now a pair of lines, whose equation, by a proper choice of co-ordinates, can be put as $x^2-z^2=0$. The polar of any point $P'(x',y',z')$ is the



line $xx'-zz'=0$ and CP$'$ is $xy'-x'y=0$, whence the inverse point P is given by—

$$x : y : z = x' : y' : x'^2/z' = x'z' : y'z' : x'^2$$

$$\therefore \quad x' : y' : z' = zx : yz : x^2 = x : y : x^2/z.$$

Hence, if $f(x, y, z)=0$ be the locus of P, that of P$'$ is

$$f(x, y, x^2/z)=0.$$

It is to be noticed that this transformation is equivalent to the three transformations in succession, in which the pole is the point C(0, 0, 1) and the bases are the three conics.

$$x^2-xy+z^2=0,\quad x^2-y^2+z^2=0 \text{ and } x^2+xy-z^2=0.$$

Case II : When the three points A, B, C coincide at C, any chord through C, the tangent at C and the tangent at the other extremity of the chord are taken as the sides of the triangle of reference.

The bas-conic is now of the form $2yz-mx^2=0$, where m is at our disposal.

The polar of P$'(x', y', z')$ is $y'z+yz'-mxx'=0$ and CP$'$ is the line $xy'-x'y=0$, whence P(x, y, z) is given by—

$$x : y : z = x'y' : y'^2 : mx'^2-y'z'$$

and $$x' : y' : z' = xy : y^2 : mx^2-yz \quad (\S\ 208).$$

219. Nöther's Transformation :

We have so long used the same triangle of reference for the curve and its transform; but if we take CBA instead of ABC as the triangle of reference for the transformed curve, this amounts simply to the interchange of x and z in the transformed equation. Hence, the curve $f(x, y, z)=0$ is transformed into $f(z, y, z^2/x)=0$.

Writing this equation in the form $f(x/z, xy/z^2, 1)=0$ we see that in the Cartesian system, the curve $f(x, y)=0$ is transformed into $f(x, xy)=0$.

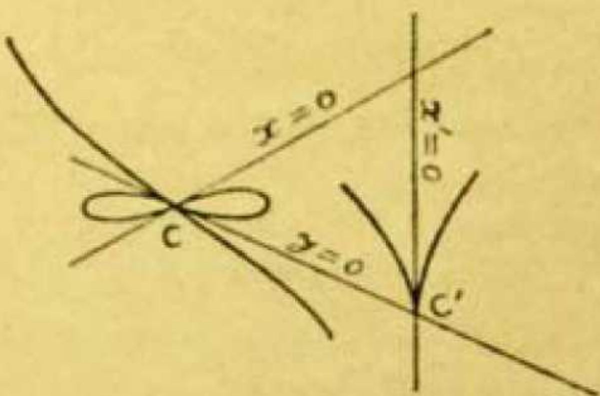
Hence, the formulæ of transformation become—

$x:y:1=x' : x'y' : 1$, *i.e.*, $x=x'$, $y=x'y'$ and $y'=y/x$, $x'=x$.

This form of transformation was given by Nöther * and was used by Newton and Cramer for the analysis of higher singularities. A series of successive transformations are at times required for complete analysis.

Ex. 1. Examine the singularity at C on the curve $y^3 = x^5$.

The inverse by the formulæ is $x'^3y'^3 = x'^5$, and consequently the proper inverse is $y'^3 = x'^2$, which has at C′ (x', y') a cusp with $x' = 0$ for tangent.



Consequently, the singularity at C on the original curve is a triple point (§ 213) whose apparent form is that of an inflexion, but the penultimate form is shown as in the figure.

Ex. 2. Verify the following :

(*i*) The inverse of a line through C or B is a line through C or B.

(*ii*) The inverse of a line is a conic through C touching AB at A.

Ex. 3. Examine the singularity at the origin on the curve $y^3 = x^4$.

220. Cremona Conditions:

As explained in §199, the general transformation $x' : y' : z' = f_1 : f_2 : f_3$ is not birational, *i.e.*, from this system it is not, in general, possible to deduce another of the form $x : y : z = F'_1 : F'_2 : F'_3$ where F'_1, F'_2, F'_3 are rational functions (polynomials) in x', y', z'.

Luigi Cremona † has investigated the conditions under which such mutual expressions are possible.

* Nöther—Über die singulären Wertsysteme einer algebraischen Function und die singulären Punkte einer algebraischen Curve—Math. Ann. Bd. 9 (1876), pp. 166-182.

† Cremona has thoroughly investigated these conditions and the theory is due to him—see his Memoir *Sulle transformazione geometriche delle figure piane*—Mem. di Bologna, Vol. II (1863) and Vol. V (1865).

For applications of Cremona transformations, see a paper by A. B. Coble in the Bull. of the Am. Math. Soc., Vol. 28 (1922), pp. 329-364, to which is appended a number of important references on the subject.

If in the one system we are given $x' : y' : z' = a : b : c$, then the corresponding points in the other are given as the intersections of the curves—

$$\frac{f_1}{a} = \frac{f_2}{b} = \frac{f_3}{c} \qquad \ldots \qquad (1)$$

Now, since f_1, f_2, f_3 are polynomials of the nth degree in (x, y, z), the number of intersections will be n^2. But if f_1, f_2, f_3 intersect in p common points, the curves (1) will evidently pass through these common points, and the remaining $n^2 - p$ points will then correspond to the given point (a, b, c).

When $p = n^2 - 1$, the curves will intersect only in *one variable* point, that is to say, all but one intersections of the curves (1) being known, the co-ordinates of the only remaining point will be determinate, and thus rational functions of $(a, b, c,)$, *i.e.*, of x', y', z', and we shall have—

$$x : y : z = F'_1 : F'_2 : F'_3$$

Hence we see that this will be a birational transformation, if the three curves f_1, f_2, f_3 have $n^2 - 1$ common points of intersection.

This again is not a sufficient condition. For, if f_1, f_2, f_3 be cubic curves having eight common points, they certainly have a *ninth* point common, and consequently there is no variable intersection. But here again, if we suppose that the cubics have one node common to all, they intersect in four other ordinary points, and since to be given a node is equivalent to three conditions, seven of their intersections are known, and therefore, only two more conditions are required to determine any curve $af_1 + bf_2 + cf_3 = 0$. Now, the common points are equivalent to eight intersections, the node counting as four. Hence, one variable point is obtained corresponding to the given point.

In fact, the system of curves $af_1+bf_2+cf_3=0$ corresponds to the system of lines $ax'+by'+cz'=0$,* and should therefore be perfectly general and must not be determinate except when $a:b:c$ are given, which is equivalent to two conditions.

Therefore, the number of conditions to be satisfied by f_1, f_2, f_3 must be at least *two* less than the number of conditions determining a curve of order n.

221. From the above considerations it follows then that n being greater than two, f_1, f_2, f_3 cannot have n^2-1 common points, for then they have another common point and no variable point of intersection. If, however, f_1, f_2, f_3 have α_1 ordinary points, α_2 double points, α_3 triple points, etc., common, such that these are equivalent to n^2-1 intersections and the number of conditions thus implied be less by 2 than the number necessary to determine a curve of order n, we obtain *one* remaining variable point of intersection corresponding to the given point and the transformation becomes rational.

Since, to be given an *r-ple* point on a curve is equivalent to $\frac{1}{2}r(r+1)$ conditions and two *n-ics* intersect in r^2 points at an *r-ple* point on each, we may state the above two conditions as follow :

$$\alpha_1+2^2\alpha_2+3^2\alpha_3+\ldots+r^2\alpha_r=n^2-1 \qquad \ldots \quad (1)$$

and $$\alpha_1+3\alpha_2+6\alpha_3+\ldots+\tfrac{1}{2}r(r+1)\alpha_r=\tfrac{1}{2}n(n+3)-2 \quad \ldots \quad (2)$$

Combining (1) and (2), we may state the second condition in a simpler form :

$$\alpha_1+2\alpha_2+3\alpha_3+\ldots\ldots+r\alpha_r=3(n-1) \qquad \ldots \quad (2')$$

Positive integral values of α_1, α_2,...satisfying the equations (1) and (2′) will then determine the transformations,

* See Montesano—Napoli Rendi, Vol. 11(3), (1905), p. 259.

provided the number of higher singularities assumed to belong to the curves does not exceed the proper limit. Cremona has tabulated all the admissible solutions, for cases up to $n=10$, of the above equations, which are often referred to as "Cremona Conditions."

For a detailed discussion of the theory, the student is referred to Cremona's Memoir and to Cayley's paper above referred to and also to his paper —"Note on the theory of the rational transformation between two planes and of special system of points," Coll. Works, Vol. VII, pp. 253-55.

222. Theorem :

Every Cremona transformation may be reduced to a number of successive quadric transformations, and conversely, each birational transformation of a plane into another is equivalent to a finite number of quadric transformations.*

Consider the transformation :

$$x' : y' : z' = f_1 : f_2 : f_3,$$

where f_1, f_2, f_3 are curves (polynomials) of order n, having a_1 ordinary points, a_2 double points, etc., in common. Then, there are three of these points† (one *q-ple*, one *r-ple* and one *s-ple*, say) the sum of whose orders exceeds n, so that

$$q+r+s>n.$$

Now, take these three points as principal points of a quadric transformation. Then the degree of the

* *Vide* Prof. Cayley's paper—"On the Rational Transformation between Two Spaces," Coll. Works, Vol. VII, pp. 189-240.

For other proofs see Nöther, Ueber Flächen, etc., Math. Ann. Bd., 3(1871), pp. 161-227 ; Segre, Un'osservazione relativa, etc., Torino Atti, Vol. 36 (1901), pp. 645-651 ; and Castelnuovo, "Le transformazioni, etc.," *ibid*, Vol. 36 (1901), pp. 861-874.

† *Vide* Salmon, H. P. Curves, § 356.

transformed curve is, by §215, $2n-q-r-s$, which is certainly less than n, *i.e.*, the degree of the given curve is reduced, and by a second quadric transformation, the degree of this curve may be further reduced. Proceeding in this way, we shall ultimately obtain right lines corresponding to the n-ics. Thus the Cremona transformation is reduced to a number of successive quadric inversions.

But, it was proved that the deficiency remains unaltered by quadric transformation, and consequently, it remains unaltered by any Cremona transformation.

223. Deficiency unaltered by Cremona Transformation:

Let $F=0$ be a curve of order k and let us apply the transformation to this curve. If now f_1, f_2, f_3 have a point A in common, the line corresponding to A will meet the transform in k points all corresponding to A, which then becomes a *k-ple* point. In general, any of the *r-ple* points becomes a *kr-ple* point. Hence, if the given curve has no multiple points, the transform will have none except at the principal points, *i.e.*, at the common points of f_1, f_2 and f_3.

Thus, the degree of the transform is nk and the corresponding maximum number of double points, as usual, is $\frac{1}{2}(nk-1)(nk-2)$. Also the multiple points at the principal points are equivalent to—

$$\tfrac{1}{2}a_1k(k-1)+\tfrac{1}{2}a_2.2k(2k-1)+\ldots+\tfrac{1}{2}a_r rk(rk-1)$$

or
$$\tfrac{1}{2}k^2(a_1+2^2a_2+3^2a_3+\ldots+r^2a_r)$$
$$-\tfrac{1}{2}k(a_1+2a_2+3a_3+\ldots+ra_r)$$

which, by equations (1) and (2′) of § 221, is equal to

$$\tfrac{1}{2}k^2(n^2-1)-\tfrac{1}{2}k.3(n-1)=\tfrac{1}{2}k^2(n^2-1)-\tfrac{3}{2}k(n-1).$$

Hence, the deficiency of the transform becomes—

$$\tfrac{1}{2}(nk-1)(nk-2)-\{\tfrac{1}{2}k^2(n^2-1)-\tfrac{3}{2}k(n-1)\}$$
$$=\tfrac{1}{2}(k-1)(k-2),$$

the same as that of the original curve.

If, however, the original curve has other multiple points, the transform will have corresponding multiple points of the same order and the deficiency will remain unaltered (§ 222). Further modification is necessary when the original curve passes through any of the principal points.

Again, when the curve $F=0$ passes through the principal points $a_1, a_2, \ldots$ the degree of the transform will be—

$$N \equiv nk-a_1-2a_2-3a_3\ldots-ra_r.$$

224. Riemann Transformation :

We have hitherto considered the Cremona transformations which are birational with regard to points of the whole plane, under certain conditions. But there are other transformations that are birational* only as regards the points of a curve of the plane, but no such conditions are necessary in this case.

Let $F=0$ be a given curve and apply the transformation

$$x' : y' : z' = f_1 : f_2 : f_3,$$

where f_1, f_2, f_3 are homogeneous functions of the nth degree in x, y, z, not necessarily satisfying Cremona's conditions, which have no common factor. The above equations are not by themselves sufficient to express x, y, z

* For the birational transformation of a curve into itself, see H. A. Schwarz, Crelle, Bd. 87 (1875), p. 189; also F. Klein, Über Riemann's Theorie der algebraischen Functionen (1882), p. 64.

rationally in terms of x', y', z'; but when they are combined with the equation $F=0$ of the curve, it is possible to express x, y, z rationally in terms of x', y', z' by the following equations:

$$x : y : z = \phi'_1 : \phi'_2 : \phi'_3$$

where ϕ'_1, ϕ'_2, ϕ'_3 are homogeneous functions in x', y', z' of the same degree n, without a common factor.

In fact, when x, y, z are eliminated between the equations of transformation and $F=0$, we obtain an equation $F'=0$, which is the condition for the co-existence of the system of equations. When this condition is satisfied, x, y, z can be determined rationally in terms of x', y', z'.*

Definition: An algebraic transformation that is birational as regards the points of two curves but not as regards the points of the whole plane is called a *Riemann Transformation*.

Ex. 1. Consider the two curves—

$$z(y^2+x^2)=x^3 \text{ and } z'y'^2=x'^3$$

both of which have the deficiency zero. We can determine a transformation which will transform the two curves one into the other.

Any point on the first can be expressed as—

$$x : y : z = (1+\lambda^2) : \lambda(1+\lambda^2) : 1$$

and any point on the second is given by—

$$x' : y' : z' = \lambda'^2 : \lambda'^3 : 1.$$

If now we associate the points of the two curves which have the same parameter, *i.e.*, $\lambda=\lambda'$, then

$$\frac{y}{x}=\frac{y'}{x'} \text{ and } \frac{z}{x}=\frac{1}{1+\lambda^2}=\frac{z'}{x'+z'},$$

whence $\quad x : y : z = x'(x'+z') : y'(x'+z') : z'x'.$

and also $\quad x' : y' : z' = x(x-z) : y(x-z) : zx.$

If, again, λ and λ' are connected by a bilinear relation of the form $A\lambda\lambda'+B\lambda+C\lambda'+D=0$, we may, in a similar manner, express $x' : y' : z'$ in terms of x, y, z.

* Salmon's Higher Algebra, Lesson X.

Ex. 2. Consider the two curves—

$$x : y : z = t^2 : t : 1+t^2$$

and $$x' : y' : z' = t(t^2-1) : (t^2-1)^2 : t.$$

Associating the points which have the same parameter, we shall show that x, y, z can be expressed rationally in terms of x', y', z', and *vice versa*.

In the first curve, $\dfrac{x}{y} = t$ and $\dfrac{x}{z} = \dfrac{t^2}{1+t^2}$ ... (1)

In the second curve, $\dfrac{x'}{y'} = t\dfrac{z'}{x'}$ and $\dfrac{x'}{z'} = t^2 - 1$ ... (2)

$$\therefore \frac{x}{y} = t = \frac{x'^2}{y'z'} \text{ and } \frac{x}{z} = \frac{t^2}{1+t^2} = \frac{x'+z'}{x'+2z'}$$

whence $x : y : z = 1 : \dfrac{y'z'}{x'^2} : \dfrac{x'+2z'}{x'+z'}$

$$= x'^2(x'+z') : y'z'(x'+z') : x'^2(x'+2z'). \qquad \text{... (A)}$$

Again, $$\frac{x'}{y'} = t.\frac{z'}{x'} = \frac{t}{t^2-1} = \frac{x}{y}\cdot\frac{1}{x^2/y^2-1} = \frac{xy}{x^2-y^2}$$

and $$\frac{x'}{z'} = t^2 - 1 = x^2/y^2 - 1 = (x^2-y^2)/y^2$$

$$\therefore x' : y' : z' = 1 : \frac{x^2-y^2}{xy} : \frac{y^2}{x^2-y^2} = xy(x^2-y^2) : (x^2-y^2)^2 : xy^3. \text{ (B)}$$

Thus, by Riemann transformation the two given curves can be transformed one into the other.

Ex. 3. Consider the curve $x^3 + y^3 + z^3 = 0$

Apply the transformation $x' : y' : z' = x^2 : y^2 : z^2$.

Now $x : y : z = 2x^4y^3z^3 : 2x^3y^4z^3 : 2x^3y^3z^4$

$$= x^4(x^6 + 2y^3z^3 - x^6) : y^4(y^6 + 2z^3x^3 - y^6)$$
$$: z^4(z^6 + 2x^3y^3 - z^6)$$
$$= x^4\{x^6 + 2y^3z^3 - (y^3+z^3)^2\} : y^4\{y^6 + 2z^3x^3 - (z^3+x^3)^2\}$$
$$: z^4\{z^6 + 2x^3y^3 - (x^3+y^3)^2\}$$
$$= x^4\{2x^6 - (x^6+y^6+z^6)\} : y^4\{2y^6 - (x^6+y^6+z^6)\}$$
$$: z^4\{2z^6 - (x^6+y^6+z^6)\}$$
$$= x'^2\{2x'^3 - k\} : y'^2\{2y'^3 - k\} : z'^2\{2z'^3 - k\}$$

where $k = x'^3 + y'^3 + z'^3$.

Thus x, y, z have been expressed rationally in terms of x', y', z' with the help of the equation of the given curve.

Now applying this transformation to the given curve, we have for the equation of the transformed, after rejecting a factor,

$$x'^6+y'^6+z'^6-2y'^3z'^3-2z'^3x'^3-2x'^3y'^3=0$$

which gives the reciprocal polar curve of $x^3+y^3+z^3=0$, and as is known, there is (1, 1) correspondence between the points and lines of two reciprocal figures.

225. Reduction of the order of the Transformed Curve :*

From what has been said before, it follows that if we apply the transformation of § 221 to the n-*ic* F, the order of the transformed curve will be $\mathrm{N}\equiv nk-a_1-2a_2\ldots$ etc., where a_1, a_2, etc., denote the number of single, double, etc., points common to the k-*ics* f_1, f_2, f_3 lying on F. We shall now consider how this transformation can be applied so as to reduce the order of the transformed curve as low as possible, *i.e.*, to make N a minimum.

Now, the curves f_1, f_2, f_3 can be made to satisfy, as has been seen in § 221, $\frac{1}{2}k(k+3)-2$ conditions.

Hence, N will be a minimum, if f_1, f_2, f_3 be assumed to pass through as many as possible of the double points of the given curve F.

If then the deficiency of F be denoted by p, the number of its double points is—

$$\tfrac{1}{2}(n-1)(n-2)-p, \quad i.e., \quad \tfrac{1}{2}n(n-3)-p+1.$$

(i) Suppose $k=n-1$.

Then, f_1, f_2, f_3, may be made to pass through

$$\tfrac{1}{2}k(k+3)-2=\tfrac{1}{2}(n-1)(n+2)-2$$

$$=\tfrac{1}{2}n(n+1)-3 \text{ points only.}$$

*Cf. Salmon, H. P. Curves, § 365.

Therefore, besides the double points, the curves f_1, f_2, f_3 can be made to pass only through

$$\{\tfrac{1}{2}n(n+1)-3\}-\{\tfrac{1}{2}n(n-3)-p+1\}$$

i.e., $2n+p-4$ ordinary points on F, so that we may take

$$a_1=2n+p-4 \quad \text{and} \quad a_2=\tfrac{1}{2}n(n-3)-p+1.$$

Therefore, the order of the transformed curve is

$$\begin{aligned} N&=nk-a_1-2a_2 \\ &=n(n-1)-(2n+p-4)-2\{\tfrac{1}{2}n(n-3)-p+1\} \\ &=p+2. \end{aligned}$$

(*ii*) Put $k=n-2$, $(n>2)$.

As before, we may take $a_2=\frac{1}{2}n(n-3)-p+1$, so that

$$\begin{aligned} a_1&=\{\tfrac{1}{2}k(k+3)-2\}-\{\tfrac{1}{2}n(n-3)-p+1\} \\ &=\{\tfrac{1}{2}(n-2)(n+1)-2\}-\{\tfrac{1}{2}n(n-3)-p+1\} \\ &=n+p-4. \end{aligned}$$

$$\begin{aligned} \therefore \quad N&=n(n-2)-a_1-2a_2 \\ &=n(n-2)-(n+p-4)-2\{\tfrac{1}{2}n(n-3)-p+1\} \\ &=p+2. \end{aligned}$$

(*iii*) Put $k=n-3$.

We take $a_2=\frac{1}{2}n(n+3)-p+1$, and consequently, $a_1=p-3$ as before, so that p is to be taken always greater than 2.

Hence $\qquad N=p+1.$

Since the transform has the same deficiency as the given curve, we may summarise the above results in the following theorem:

A curve of order n with deficiency p may be transformed into a curve of order $p+2$ with deficiency p or with $\frac{1}{2}p(p-1)$ double points.

If $p>2$, the order of the transform may be $p+1$ with deficiency p, or with $\frac{1}{2}p(p-3)$ double points.

If, however, $p=0$, the curve may be transformed into a conic, which however can be further transformed into a straight line.

If $p=1$, the transform is a cubic, and so on.

For a detailed discussion, the student is referred to Brill-Nöther's paper—"Ueber die algebraichen Funcktionen and ihre Anwendung in der Geometrie"—Math. Ann. Bd. 7, pp. 297-398, and also to Cayley's paper—"On the Transformation of Plane Curves," Coll. Works, Vol. 6, pp. 1-9.

226. Reduction of a Curve with Multiple Points :

The following formal proof for the general case of a curve with multiple points was given by Scott.*

Let F have multiple points of orders r_1, r_2, r_3, etc., and at these points let the curves f_1, f_2, f_3 have multiple points of orders ρ_1, ρ_2, ρ_3 ... etc. (where any of the r's or ρ's may be zero or unity).

It is required to determine k and ρ's, so that the order of the transformed curve $N=nk-\Sigma r_i\rho_i$ may be a minimum, *i.e.*, for a given value of k, $\Sigma r_i\rho_i$ is to be made a maximum.

The curves f_1, f_2, f_3 can be made to satisfy $\frac{1}{2}k(k+3)-2$ conditions only, but if a ρ-point of the f's is placed at an ordinary point of F, the number of conditions imposed is $\frac{1}{2}\rho(\rho+1)$, while the point counts as ρ intersections.

Evidently $\frac{1}{2}\rho(\rho+1)\gtreqless\rho$, according as $\rho\gtreqless 1$. Hence, an ordinary point on F will count as most intersections, if it be an ordinary point of f's, *i. e.*, if $\rho=1$.

Again, if a ρ-point is placed at an r-point, the number of conditions is $\frac{1}{2}\rho(\rho+1)$, while the number of intersections is $r\rho$, and $r\rho-\frac{1}{2}\rho(\rho+1)$ is certainly a positive quantity, if $r>1$ and $\rho=1$, and generally, the difference between the number of intersections and the number of conditions is to be made a maximum.

Since the multiple points are supposed independent, the existence of other multiple points will not affect the number of conditions imposed upon f_1, f_2, f_3, by supposing the ρ-point at the *r-ple* point of F. Hence

* Scott, "Note on Adjoint Curves," Quarterly Journal of Math., Vol. XXVIII, pp. 377-381.

at every *r-ple* point of F we have to make the difference a maximum *i.e.*, to make $r\rho-\frac{1}{2}\rho(\rho+1)$ or, $\frac{1}{2}\rho(2r-\rho-1)$ a maximum.

Now the sum of the two factors being given, the product will be a maximum when the two factors are as nearly equal as possible, *i.e.*, when the factors are r and $r-1$. Thus we may take $\rho=r$ or $=r-1$, and in either case the above expression $=\frac{1}{2}r(r-1)$.

Hence, at every *r-ple* point of F we may take $\rho=r$, or $\rho=r-1$, and then take other σ ordinary points on F sufficient to make up the number of conditions necessary for the transformation-net.

But since an *r-ple* point of f's may be regarded as an $(r-1)$*-ple* point with r of the σ other conditions, the case $\rho=r$ may be included in the case $\rho=r-1$. Hence we may summarise the result as follows :—

The reduction of the order of a general curve as lowest as possible must be effected by means of a transformation in which the transformation-curves shall have an $(r-1)$*-ple point at every r-ple point on F, and shall pass through other* σ *ordinary points on F, sufficient to make up the number of conditions required for the net,* or, in other words, *the transformation must be effected by means of Adjoint Curves.*

Definition :

An "adjoined" or "adjoint" curve is one which has an $(r-1)$*-ple* point at every *r-ple* point of an *n-ic*.

We shall next show that best results can be obtained, in general, by using adjoints of order $k=n-3$, *i.e.*, we shall find the value of k which will minimise $N=nk-\Sigma r_i\rho_i-\sigma$, where $\rho=r-1$, and σ gives the number of ordinary points on F that may be chosen arbitrarily for the determination of f_1, f_2, f_3.

But there is a limit to the value of σ. For the number of points on an *n-ic* of deficiency p which can be chosen to determine a curve of lower order k is $nk-p$, and the remaining points are thereby determined, if $k\geq n-2$. But if $k\leq n-3$, there is no such limitation. Hence, for the net of adjoints, if $k\geq n-2$, the number of intersections used in imposing conditions falls short of the total number by $p+2$, which shows that $\Sigma r(r-1)+\sigma=nk-p-2$

$$\therefore \quad N=nk-\Sigma r(r-1)-\sigma=p+2, \quad \text{when } k\geq n-2.$$

If, $k\leq n-3$, we have $\frac{1}{2}\Sigma r(r-1)+\sigma=\frac{1}{2}k(k+3)-2$

i.e., $$\sigma=\tfrac{1}{2}k(k+3)-\tfrac{1}{2}(n-1)(n-2)+p-2$$

which implies that the expression must not be negative.

Now, writing $k=n-3+t$, we have $2\sigma=t^2-6n+2nt-3t-6+2p$, which shows that $2p\leq 6-t(2n-3+t)$, *i.e.*, the value of n depends upon n, which therefore is a special case and is not to be considered.

$\therefore$ t must be zero for the general case, and hence $k=n-3$, $\sigma=p-3$ and $N=p+1$, subject to the condition $p \nless 3$.

Thus, in general, the lowest possible order of the transformed curve is obtained by means of adjoints of order $n-3$, passing through certain ordinary points of the given curve.

The question of further reduction of the order, by a proper choice of the ordinary points, should be discussed separately.

227. Adjoint Curves:

From what has been said above about transformation curves, it is seen that the special class, called "adjoints", which is so familiar to the student of function-theory, plays no unimportant a part in the geometry of plane algebraic curves. As we have seen, the adjoints of order lower by three than the original curve are important from the fact that they always transform into corresponding adjoints of the transform curves. Adjoints to the adjoints of a curve are called *Second Adjoints*, and so on. The use of successive adjoints as a means of investigation is due to S. Kantor and G. Castelnuovo.*

Now, the fact that a curve has $(r-1)$-*ple* point at a given point is equivalent to $\frac{1}{2}r(r-1)$ conditions (§ 50). Consequently, the co-efficients in the equation of the adjoint $(n-3)$-*ic* must be connected by $\Sigma\frac{1}{2}r(r-1)$ relations extending over all the multiple points of the n-*ic*, and its equation therefore contains $\frac{1}{2}n(n-3)-\Sigma\frac{1}{2}r(r-1)$ arbitrary co-efficients.

But $\frac{1}{2}n(n-3)-\Sigma\frac{1}{2}r(r-1)=\{\frac{1}{2}(n-1)(n-2)-\Sigma\frac{1}{2}r(r-1)\}-1$
$$=p-1 \text{ (§ 53).}$$

Thus we may state that the deficiency of an n-*ic* is one more than the number of arbitrary co-efficients in the equation of the most general adjoint $(n-3)$-*ic*.

* Math. Ann. Bd. 44 (1894), p. 127.

It is to be noticed, however, that for $n=1$ or 2, $p=0$; and for $n=3$, $p=0$ or 1, according as the *n-ic* (cubic) has or has not a double point.

228. Intersection of a Curve with its Adjoint:

Since at every *r-ple* point the adjoint has an $(r-1)$-*ple* point, the point counts as $r(r-1)$ intersections, and the fact that the adjoint has an $(r-1)$-*ple* point is equivalent to $\frac{1}{2}r(r-1)$ relations between its co-efficients.

Hence we obtain the theorem:

The number of intersections of an **n**-*ic and an adjoint at the multiple points of the n-ic is double the number of relations between the co-efficients of the adjoint curve.*

If the adjoint be an $(n-3)$-*ic*, since there are $p-1$ arbitrary co-efficients, the number of relations between its co-efficients is $\frac{1}{2}n(n-3)-p+1=\frac{1}{2}(n-1)(n-2)-p$.

Ex. Show that the identity (1) of § 38 holds, if C_m, C_m', C_l, C_l' are adjoints to C_n.

229. Intersections with a Pencil of Adjoints:

Let k be the order of a curve adjoint to the *n-ic*, with multiple points of orders $r_1, r_2, r_3, \ldots$ Then the multiple points count as $\Sigma r(r-1)$ intersections and the co-efficients of the adjoint *k-ic* are connected by $\frac{1}{2}\Sigma r(r-1)$ relations.

Therefore the *k-ic* requires $\frac{1}{2}k(k+3)-\frac{1}{2}\Sigma r(r-1)$ other conditions to be uniquely determined, *i.e.*, we may take $\frac{1}{2}k(k+3)-\frac{1}{2}\Sigma r(r-1)$ other ordinary points on the *n-ic* besides the multiple points, so as to completely determine the adjoint.

Now, the two curves intersect in nk points. Hence the number of remaining intersections

$$\begin{aligned}
&=nk-\Sigma r(r-1)-\{\tfrac{1}{2}k(k+3)-\tfrac{1}{2}\Sigma r(r-1)\}\\
&=nk-\tfrac{1}{2}\Sigma r(r-1)-\tfrac{1}{2}k(k+3)\\
&=nk+p-\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}k(k+3)\\
&=\tfrac{1}{2}(2nk-n^2-k^2+3n-3k)+p-1\\
&=\tfrac{1}{2}(n-k)(k-n+3)+p-1.
\end{aligned}$$

This result shows that if we describe a pencil of k-ics through the multiple points and through

$$\{\tfrac{1}{2}k(k+3)-1\}-\tfrac{1}{2}\Sigma r(r-1)$$

other ordinary points on the n-ic, then this pencil will meet the n-ic in $\frac{1}{2}(n-k)(k-n+3)+p$ variable points.

Hence, we may state the theorem :

Any curve of a pencil of adjoint k-ics, through the multiple points and other ordinary fixed points on the n-ic, will meet the n-ic in $\frac{1}{2}(n-k)(k-n+3)+p$ *variable points.*

If $k=n-1$ or $n-2$, this number is $p+1$; if $k=n-3$, it is equal to p.

Thus, any adjoint $(n-3)$-ic through the multiple points and through $\frac{1}{2}n(n-3)-1-\frac{1}{2}(n-1)(n-2)+p$, *i.e.*, $p-2$ ordinary points on the n-ic will meet the n-ic in p other variable points.

Ex. A pencil of adjoint k-ics has its base-points on an n-ic. Show that $(n-k)(k-n+3)+4p-2$ curves of the pencil touch the n-ic at points other than a base-point.

230. Transformation by Adjoints :

Let there be a_1 double points, a_2 triple points, and a_r $(r+1)$-*ple* points on the given n-*ic* $F=0$, so that the adjoint k-*ics* have common a_1 single points, a_2 double points, ...a_r r-*ple* points on F.

If p' denotes the deficiency of the adjoints, by § 53.

$$p=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\Sigma r(r+1)a_r \qquad \dots \quad (1)$$

$$p'=\tfrac{1}{2}(k-1)(k-2)-\tfrac{1}{2}\Sigma r(r-1)a_r \qquad \dots \quad (2)$$

Now, $$p=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\Sigma r(r-1)a_r-(a_1+2a_2+\dots+ra_r)$$

$$=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\Sigma r(r-1)a_r-3(k-1). \quad (\S\ 221.)$$

$$\therefore\quad p'-p=\{\tfrac{1}{2}(k-1)(k-2)-\tfrac{1}{2}\Sigma r(r-1)a_r\}$$

$$-\{\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\Sigma r(r-1)a_r-3(k-1)\}$$

$$=\tfrac{1}{2}(k-1)(k-2)-\tfrac{1}{2}(n-1)(n-2)+3(k-1)$$

i.e., $$p'=\tfrac{1}{2}(k-1)(k-2)-\tfrac{1}{2}(n-1)(n-2)+3(k-1)+p.$$

Hence, if $k=n-1$, $p'=2n+p-4=$ number of ordinary points.

If $k=n-2$, $p'=n+p-4=$ number of ordinary points.

If $k=n-3$, $p'=p-3=\sigma=$ number of ordinary intersections. Since it is a (1, 1) correspondence, there should be no ordinary intersection on F.

$\therefore\ \sigma=p-3=p'=0$, *i.e.*, the adjoints must be unicursal.

That this is a sufficient condition can be shown as follows :

As before, $$\tfrac{1}{2}\Sigma r(r+1)\alpha_r=\tfrac{1}{2}(n-1)(n-2)-p \quad \ldots \quad (3)$$

and $$\tfrac{1}{2}\Sigma r(r-1)\alpha_r=\tfrac{1}{2}(k-1)(k-2)-p' \quad \ldots \quad (4)$$

and if q be the number of free intersections besides the multiple and other points on the given curve, then

$$q=k^2-\Sigma r^2\alpha_r-\sigma \quad \ldots \quad (5)$$

(*i*) If $k=n-3$, $\sigma=p-3$, and (4) can be written as

$$\tfrac{1}{2}\Sigma r(r-1)\alpha_r=\tfrac{1}{2}(n-4)(n-5)-p' \quad \ldots \quad (6)$$

From (1) and (6) we have $\Sigma r^2\alpha_r=n^2-6n+11-p-p'$

$$\therefore\ q=(n-3)^2-(n^2-6n+11-p-p')-(p-3)=p'+1.$$

(*ii*) If $k>n-3$, we have $\sigma=nk-p-2-\Sigma r(r+1)\alpha_r$. Substituting this value of σ and that of $\Sigma r^2\alpha_r$ obtained by addition of (1) and (2) in equation (5), we obtain—

$$q=k^2-nk-\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}(k-1)(k-2)+p'+2$$

$$=\tfrac{1}{2}(n-k)(n-k-3)+p'+2$$

$$=\tfrac{1}{2}(n-k-1)(n-k-2)+p'+1.$$

$\therefore$ If $k=n-1$ or $n-2$, $q=p'+1$, the same as above.

Hence, by means of adjoints of order $\angle n$, but $\nless n-3$, and deficiency p', the plane is subjected to a $(q, 1)$, *i.e.*, $(p'+1, 1)$ transformation.

$\therefore$ It will be a (1, 1) Cremona transformation, if $p'=0$, *i.e.*, if the adjoints are unicursal.

Thus the necessary and sufficient condition that the transformation by adjoints is a (1, 1) transformation is that the adjoints be unicursal.

It is to be noticed, however, that the number of free intersections of a net of curves (not necessarily adjoints) of deficiency p', passing in a specified manner through fixed points is $p'+1$, and this is a particular case of a theorem due to Segre.*

* Segre, Rendiconti del Circolo Matematico di Palermo, Vol. 1 (1887), p. 217.

CHAPTER XI

Unicursal Curves

231. Parametric Representation:

It has already been said that a curve is rational or unicursal when its deficiency is zero, and conversely. In the present Chapter, however, will be discussed certain properties of all curves, the co-ordinates of whose points can be expressed in terms of a single variable parameter.

The real significance of the two co-ordinates of a point in the Cartesian system lies in the fact that a point has two degrees of freedom in the plane, and is practically contained in the two statements:

(1) The point lies on a certain locus;

(2) The point has a particular position on this locus.

Hence, the homogeneous co-ordinates must be expressible in terms of two independent parameters in the form

$$x : y : z = f_1(\lambda, \mu) : f_2(\lambda, \mu) : f_3(\lambda, \mu)$$

where f_1, f_2, f_3 represent rational, integral functions of order n without a common factor.

When one of the parameters (μ) is given, one degree of freedom is lost, and the point is restricted to lie on the locus $\mu = const.$ and the co-ordinates * are defined by

$$x : y : z = f_1(\lambda) : f_2(\lambda) : f_3(\lambda)$$

where λ is a variable parameter. The elimination of λ from these equations leaves the position of the point on the curve undetermined, but still the point lies on the curve, and we obtain the equation of the curve.†

* Brill, Math. Ann. Bd. 5 (1872), p. 401. See also the papers by Luroth, *ibid*, Bd. 9, Pasch, Bd. 18, and Humbert, Bull. Soc. Math. de France *t*. 13 (1885), pp. 49 and 89.

† *Cf.* J. E. Rowe, Bull. Am. Math. Soc., Vol. 23 (1917), pp. 304-308.

232. Clebsch Method:

If the n-*ic* is of zero deficiency, it has $\frac{1}{2}(n-1)(n-2)$ double points. Through these double points and through $\frac{1}{2}k(k+3)-\frac{1}{2}(n-1)(n-2)$ other points on the curve $(k<n)$, a curve of order k can be drawn, which intersects the n-*ic* in only kn points. Since each double point counts as *two* intersections, we must have

$$\tfrac{1}{2}k(k+3)+\tfrac{1}{2}(n-1)(n-2)=kn$$

which requires that $\quad k=n-1$ or $k=n-2$.

Therefore, if we construct a pencil of $(n-1)$-*ics* $u-\lambda v=0$ through the double points and through $2n-3$ other points on the given n-*ic*, then each curve of the pencil will intersect the n-*ic* in one point whose co-ordinates can be rationally expressed in terms of λ.

In a similar manner, if a pencil of $(n-2)$-*ics* be drawn through the double points and $n-3$ ordinary points on the n-*ic*, each curve will intersect the latter in a point whose co-ordinates can be similarly expressed.

Thus, by taking a pencil of $(n-1)$-*ics* or $(n-2)$-*ics* we can express the co-ordinates rationally in terms of a parameter.

The above considerations at once suggest that there always exists a k-*ic* which intersects the unicursal n-*ic* in a number of fixed points, or has assigned singularities at those points, such that the points count as $nk-1$ intersections.

Case I: The n-*ic* has no multiple points other than ordinary nodes and cusps.

If $k=n-1$, the co-efficients in the equation of the k-*ic* are connected by $\frac{1}{2}(n-1)(n-2)+2n-3=\frac{1}{2}k(k+3)-1$ linear relations, and the intersections count as—

$$2\times\tfrac{1}{2}(n-1)(n-2)+(2n-3)=nk-1 \text{ points.}$$

Case II: The *n-ic* has ordinary multiple points and the *k-ic* has an $(r-1)$-*ple* point at each *r-ple* point on the *n-ic*, and passes through $(n-3)$ other fixed points.

If $k=n-2$, since $p=\frac{1}{2}(n-1)(n-2)-\Sigma r(r-1)=0$, the co-efficients in the equation of the *k-ic* are connected by

$$\Sigma\tfrac{1}{2}r(r-1)+(n-3)=\tfrac{1}{2}(n-1)(n-2)+(n-3)$$
$$=\tfrac{1}{2}k(k+3)-1 \quad (\S\ 50)$$

linear relations, and the intersections count as—

$$\Sigma r(r-1)+(n-3)=(n-1)(n-2)+(n-3)$$
$$=nk-1 \text{ points.}$$

In fact, the *k-ic* is an adjoint $(n-2)$-*ic* passing through $(n-3)$ ordinary points on the *n-ic*.

233. The Order of the Unicursal Curve:

Consider the curve—

$$x:y:z=f_1(\lambda):f_2(\lambda):f_3(\lambda) \quad \ldots \quad (1)$$

where f_1, f_2, f_3 are rational, integral, homogeneous functions of order n in λ. To each point of the curve there corresponds a certain value of the parameter λ, and conversely. The functions f_1, f_2, f_3 cannot have any common factor.

Any line $lx+my+nz=0$ intersects the curve in n points given by—

$$lf_1+mf_2+nf_3=0 \quad \ldots \quad (2)$$

and consequently, its order is n.

That the equations (1) represent a general curve of order n, having $\frac{1}{2}(n-1)(n-2)$ double points, follows from the fact that the expressions (1) contain $3(n+1)$ arbitrary constants. By a linear transformation of the form—

$$\lambda=\frac{p\lambda'+q}{r\lambda'+s}$$

we may reduce the number to $3n-1$.

But $\qquad 3n-1=\frac{1}{2}n(n+3)-\frac{1}{2}(n-1)(n-2)$

i.e., equal to the number required for determining a

general curve of order n with $\frac{1}{2}(n-1)(n-2)$ double points, which proves the proposition.

234. The Class of the Unicursal Curve: *

The equation of the curve in line co-ordinates is obtained by forming the discriminant of the equation (2), *i.e.*, by eliminating λ between the equations

$$\left.\begin{aligned} \frac{\partial f_1}{\partial\lambda}+m\frac{\partial f_2}{\partial\lambda}+n\frac{\partial f_3}{\partial\lambda}=0 \\ lf_1(\lambda)+mf_2(\lambda)+nf_3(\lambda)=0 \end{aligned}\right\} \quad \dots \quad (3)$$

The discriminant is, in general, of degree $2(n-1)$, and consequently, the curve is of class $2(n-1)$.

If, however, $\frac{\partial f_1}{\partial\lambda}:\frac{\partial f_2}{\partial\lambda}:\frac{\partial f_3}{\partial\lambda}=f_1:f_2:f_3 \quad \dots \quad (4)$

the two equations become identical, and the common roots of (4) reduce the class of the curve.

Hence, if the equations (4) have κ common roots, the class of the curve is $m=2(n-1)-\kappa$. We shall see later on that the equations (4) give the parameters of cusps, and hence, if there are κ cusps, the class is $m=2(n-1)-\kappa$.

235. Parametric Representation in Line Co-ordinates:

By eliminating l, m, n between the equations (3) and $lx+my+nz=0$, we obtain the equation of the tangent at any point in the form—

$$\begin{vmatrix} x & y & z \\ \frac{\partial f_1}{\partial\lambda} & \frac{\partial f_2}{\partial\lambda} & \frac{\partial f_3}{\partial\lambda} \\ f_1 & f_2 & f_3 \end{vmatrix}=0 \quad \dots \quad (5)$$

* Clebsch, Crelle, Bd. 64 (1865), p. 43. See also Hasse, Math. Ann. Bd. 2 (1870), p. 515.

Hence, the co-ordinates of the tangent are given by—

$$\rho\xi = f_3 \frac{\partial f_2}{\partial \lambda} - f_2 \frac{\partial f_3}{\partial \lambda}$$

$$\rho\eta = f_1 \frac{\partial f_3}{\partial \lambda} - f_3 \frac{\partial f_1}{\partial \lambda}$$

$$\rho\zeta = f_2 \frac{\partial f_1}{\partial \lambda} - f_1 \frac{\partial f_2}{\partial \lambda},$$

i.e., the co-ordinates of the tangent are rational, integral, algebraic functions of a parameter. Hence we obtain the theorem :

If a curve is unicursal quà locus, it is unicursal quà envelope.

It is to be noted here that the parameters in the co-ordinates of the tangent cannot occur in degree higher than $2(n-1)$, but may be less. It follows then that the class of the curve cannot be greater than $2(n-1)$.

Ex. 1. Consider the parabola $y^2 = 4ax$.

The co-ordinates of any point on this curve can be expressed in the form $x = at^2, \quad y = 2at$.

The equation of the tangent at any point (t) being $ty - x - at^2 = 0$, the co-ordinates of the tangent are $\xi = -1$, $\eta = t$, $\zeta = -at^2$, which shows that the parabola is unicursal quà envelope. The tangential equation of the curve is easily obtained in the form $a\eta^2 = \xi\zeta$.

Ex. 2. Express the co-ordinates of any point on the curve

$$(x+y)^3 z^2 = x^3 (y+z)^2$$

rationally in terms of a parameter.

Ex. 3. Prove that the order of a unicursal curve of class m is not greater than $2(m-1)$.

Ex. 4. The locus of the poles of any normal to a given conic is, in general, a unicursal quartic. Discuss the case when the conic is a parabola.

Ex. 5. If f_1, f_2, f_3 are polynomials in t of degree n, in which the co-efficient of t^{n-1} is wanting, the sum of the parameters of points where any algebraic curve intersects the n-*ic* is zero.

236. Singular Points:

As already shown, at a double point the co-ordinates have the same values for two *different* values of the parameters λ and λ', *i.e.*, when $\lambda \neq \lambda'$,

$$f_1(\lambda)/f_1(\lambda') = f_2(\lambda)/f_2(\lambda') = f_3(\lambda)/f_3(\lambda') \quad \dots \quad (1)$$

The case of a node has been discussed in § 56, but in the case of a cusp, the equations of the tangents for the two branches meeting at the double point must be the same, *i.e.*, the co-ordinates of the two tangents must be proportional.

$$\therefore \quad (f_3 \partial f_2/\partial\lambda - f_2 \partial f_3/\partial\lambda) = k'(f_3 \partial f_2/\partial\lambda' - f_2 \partial f_3/\partial\lambda')$$
$$(f_1 \partial f_3/\partial\lambda - f_3 \partial f_1/\partial\lambda) = k'(f_1 \partial f_3/\partial\lambda' - f_3 \partial f_1/\partial\lambda')$$
$$(f_2 \partial f_1/\partial\lambda - f_1 \partial f_2/\partial\lambda) = k'(f_2 \partial f_1/\partial\lambda' - f_1 \partial f_2/\partial\lambda')$$

where λ, λ' are the parameters of the coincident points belonging to the two branches, whence we obtain the two following determinant equations:

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial\lambda'} & \dfrac{\partial f_1}{\partial\lambda} & f_1(\lambda) \\ \dfrac{\partial f_2}{\partial\lambda'} & \dfrac{\partial f_2}{\partial\lambda} & f_2(\lambda) \\ \dfrac{\partial f_3}{\partial\lambda'} & \dfrac{\partial f_3}{\partial\lambda} & f_3(\lambda) \end{vmatrix} = 0, \quad \begin{vmatrix} f_1(\lambda') & \dfrac{\partial f_1}{\partial\lambda} & f_1(\lambda) \\ f_2(\lambda') & \dfrac{\partial f_2}{\partial\lambda} & f_2(\lambda) \\ f_3(\lambda') & \dfrac{\partial f_3}{\partial\lambda} & f_3(\lambda) \end{vmatrix} = 0 \quad \dots \quad (2)$$

which show nothing but that λ, λ' must be the double roots of the equation giving the nodes.

There are then two cases to be considered:

(1) When λ and λ' are different; there is no cusp but two nodes indefinitely near each other, such that their tangents coincide, and there is no reduction in the class.

(2) When λ and λ' are equal and represent the pair of equal roots, which then give a cusp, and the class of the curve is reduced by one.

Putting $\lambda' = \lambda + \epsilon$, where ϵ is a very small quantity approaching zero, from (2) we obtain—

$$\Delta \equiv \begin{vmatrix} \dfrac{\partial^2 f_1}{\partial\lambda^2} & \dfrac{\partial f_1}{\partial\lambda} & f_1 \\ \dfrac{\partial^2 f_2}{\partial\lambda^2} & \dfrac{\partial f_2}{\partial\lambda} & f_2 \\ \dfrac{\partial^2 f_3}{\partial\lambda^2} & \dfrac{\partial f_3}{\partial\lambda} & f_3 \end{vmatrix} = 0 \qquad \dots \quad (3)$$

the double roots of which give the parameters of the cusps.

Thus the cusps are given by the equations $\Delta = 0$ and

$$\frac{\partial\Delta}{\partial\lambda} \equiv \begin{vmatrix} f_1 & \dfrac{\partial f_1}{\partial\lambda} & \dfrac{\partial^3 f_1}{\partial\lambda^3} \\ f_2 & \dfrac{\partial f_2}{\partial\lambda} & \dfrac{\partial^3 f_2}{\partial\lambda^3} \\ f_3 & \dfrac{\partial f_3}{\partial\lambda} & \dfrac{\partial^3 f_3}{\partial\lambda^3} \end{vmatrix} = 0 \qquad \dots \quad (4)$$

Second Method :

The cusps may also be determined from the equations—

$$\frac{\partial f_1}{\partial\lambda} \Big/ f_1 = \frac{\partial f_2}{\partial\lambda} \Big/ f_2 = \frac{\partial f_3}{\partial\lambda} \Big/ f_3 \qquad \dots \quad (5)$$

For, at a node we have equations (1) satisfied, and putting each ratio equal to k, we have—

$$\frac{f_1(\lambda') - f_1(\lambda)}{(\lambda' - \lambda) f_1(\lambda)} = \frac{f_2(\lambda') - f_2(\lambda)}{(\lambda' - \lambda) f_2(\lambda)} = \frac{f_3(\lambda') - f_3(\lambda)}{(\lambda' - \lambda) f_3(\lambda)}$$

$$= \frac{k-1}{(\lambda' - \lambda)}.$$

But, by the mean value theorem, we have—

$$\frac{f_1(\lambda') - f_1(\lambda)}{\lambda' - \lambda} = f'_1(\lambda + \theta[\lambda' - \lambda])$$

where θ is a positive proper fraction.

Now, when the tangents at the node approach coincidence, λ' approaches λ, and we have in the limit—

$$\frac{\partial f_1}{\partial\lambda} \Big/ f_1(\lambda) = \frac{\partial f_2}{\partial\lambda} \Big/ f_2(\lambda) = \frac{\lambda f_3}{\partial\lambda} \Big/ f_3(\lambda).$$

237. Inflexions:

If λ, λ', λ'' be the parameters of three collinear points on the curve, we have—

$$lf_1(\lambda) + mf_2(\lambda) + nf_3(\lambda) = 0$$

$$lf_1(\lambda') + mf_2(\lambda') + nf_3(\lambda') = 0$$

$$lf_1(\lambda'') + mf_2(\lambda'') + nf_3(\lambda'') = 0$$

where λ, λ', λ'' are all different.

$\therefore$ The condition of collinearity is obtained by eliminating l, m, n from the above equations in the form—

$$\frac{1}{(\lambda-\lambda')(\lambda'-\lambda'')(\lambda''-\lambda)} \begin{vmatrix} f_1(\lambda) & f_2(\lambda) & f_3(\lambda) \\ f_1(\lambda') & f_2(\lambda') & f_3(\lambda') \\ f_1(\lambda'') & f_2(\lambda'') & f_3(\lambda'') \end{vmatrix} = 0$$

If now λ, λ', λ'' approach equality, but still distinct, the above condition reduces to—

$$\frac{\Delta}{2n(n-1)^2} = 0$$

Hence, the inflexions are given by the equation—

$$\Delta = 0 \qquad \ldots \quad (6)$$

Thus at a cusp, we have both $\Delta = 0$ and $\partial\Delta/\partial\lambda = 0$, while at a point of inflexion only $\Delta = 0$.

From equation (6) it follows that the number of inflexions is, in general, $3(n-2)$. But since a double root of (6) gives a cusp, the number of inflexions reduces to $3(n-2)-2\kappa$, if it has κ double roots, *i.e.*, if the curve has κ cusps.

In fact, $\quad \iota = 3n(n-2) - 3(n-1)(n-2) - 2\kappa$

$$= 3(n-2) - 2\kappa$$

This number gives an upper limit to the number of cusps which a curve can possess. For, since $3(n-2) - 2\kappa$ can never be negative, the number of cusps on a curve can never exceed $\frac{3}{2}(n-2)$.*

238. Bitangents of Unicursal Curves:

Bitangents and stationary tangents can be found exactly in the same manner, if the curve is regarded as unicursal quà envelope.

As proved in § 235, the co-ordinates of a tangent can be expressed in terms of a parameter in the form:

$$\xi : \eta : \zeta = \phi_1(\lambda) : \phi_2(\lambda) : \phi_3(\lambda)$$

Therefore, the parameters of the bitangents are obtained from

$$\frac{\phi_1(\lambda)}{\phi_1(\lambda')} = \frac{\phi_2(\lambda)}{\phi_2(\lambda')} = \frac{\phi_3(\lambda)}{\phi_3(\lambda')}, \quad \text{when } \lambda \neq \lambda'.$$

The number of bitangents of a unicursal curve is found to be $2(n-2)(n-3)$, as can be verified from the formulæ of § 146.

The parameters of inflexional tangents are given by—

$$\frac{\partial \phi_1}{\partial \lambda} \Big/ \phi_1 = \frac{\partial \phi_2}{\partial \lambda} \Big/ \phi_2 = \frac{\partial \phi_3}{\partial \lambda} \Big/ \phi_3, \qquad \text{and so on.}$$

* This was first discovered by Clebsch—Crelle's Journal (1864), p. 51. There is a further limit to this. The inflexions on an n-*ic* being given by the formula $3n(n-2) - 6\delta - 8\kappa$, for an n-*ic*, in general, $6\delta + 8\kappa \leq 3n(n-2)$. If then all the double points are cusps, *i.e.*, if $\delta = 0$, $\kappa \leq \frac{3}{8}n(n-2)$; *i.e.*, $\frac{3}{8}n(n-2)$ is an upper bound to κ, but this, in fact, cannot actually be attained. The exact upper limit to κ is given as $\frac{1}{3}n(n-2)$ by S. Lefschetz—Transactions of the Am. Math. Soc., Vol. 14 (1913), pp. 13-41

For a detailed discussion of the bitangents, etc., of a unicursal curve, the student is referred to the paper of Clebsch—Crelle, Bd. 64 (1865), pp. 53-54, and of Haase—Math. Ann., Bd. 2 (1870), p. 515.

Ex. 1. Consider the curve : $x=t^4,\ y=1+t^2,\ z=t.$

For double points, we must have—

$$t'^4=t^4,\quad 1+t'^2=1+t^2,\quad t'=t,\qquad \text{when } t\neq t'.$$

As in § 56, the nodes will be given by—

$$\phi_1=tt'-1=0 \quad \text{and} \quad \phi_2=tt'(t'^2+tt'+t^2)=0$$

Whence eliminating t' we obtain $t^4+t^2+1=0$ which gives the parameters for the nodes.

But, $$(t^4+t^2+1)=(t^2+t+1)(t^2-t+1)=0$$

whence, $t^2+t+1=0$ giving $t=\omega,\ \omega^2$, the imaginary cube roots of unity, and $t^2-t+1=0$ giving $t=\frac{1}{2}(+1\pm i\sqrt{3})$

i.e., the values of the parameters are imaginary.

The first gives the acnode $(1,-1,1)$, and the second gives the acnode $(-1,1,1)$. The cusps and inflexions are given by the determinant equation (4), § 236, which reduces to $t^2(t^2-2)=0$, whose roots are

$$t=0,\quad 0,\quad \pm\sqrt{2}$$

The two 0 values of t give, in fact, a point of undulation at $(0,1,0)$. The cusps are given by the double roots, *i.e.*, the common roots of this and $t(t^2-1)=0$. Hence $t=0$ is a common root and should give a cusp, but this is found to be a point of undulation. Therefore $(0,1,0)$ is not strictly speaking a cusp. If, however, we replace t by $1/t'$ in the original values of x, y and z, we find $x=1/t'^4,\ y=(1+t'^2)/t'^2,\ z=1/t'$.

Proceeding as before, the cusps are given by the double roots of $t'^2(1-2t'^2)=0$. Hence $t'=0$, *i.e.*, $t=\infty$ gives a cusp $(1,0,0)$.

Any line meeting the curve in points t and t' is—

$$(1-tt')x+y(t^2t'^2+t^3t'+t'^3t)-z(t^3t'^2+t^2t'^3+t^3+t^2t'+tt'^2+t'^3)=0,$$

whence, by putting $t=t'$, the equation of the tangent at any point (t) is obtained in the form—

$$(1-t^2)x+3t^4y-2t^3(t^2+2)z=0 \qquad \dots \quad (1)$$

The tangent at any other point (t') is—

$$(1-t'^2)x+3t'^4y-2t'^3(t'^2+2)z=0 \qquad \dots \quad (2)$$

We have a bitangent when (1) and (2) represent the same line, *i.e.*, if

$$\frac{1-t^2}{1-t'^2}=\frac{t^4}{t'^4}=\frac{t^3(t^2+2)}{t'^3(t'^2+2)}, \quad \text{when } t\neq t'$$

whence we get $(t+t')(t^2+t'^2-t^2t'^2)=0$ and $tt'=2$

$\therefore$ The bitangents are given by—

$$\left.\begin{matrix} t+t'=0 \\ tt'=2 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} t^2+t'^2-t^2t'^2=0 \\ tt'=2. \end{matrix}\right.$$

The first group gives $t=-t'=i\sqrt{2}$, *i.e.*, $t^2+2=t'^2+2=0$.

$\therefore$ The bitangent is $\qquad x+4y=0.$

The other group gives equal values of t and t', and hence there is no bitangent.

Bitangents may be obtained otherwise as follows :

The co-ordinates of the point of intersection of (1) and (2) are given by—

$$\frac{x}{6v^3(v-2)}=\frac{y}{2u^4+(4+2v^2-6v)(u^2-2v)+4v^2}=\frac{z}{3\{u^3-2uv-3v^2u\}}$$

where $t+t'=u$ and $tt'=v$.

If the tangents coincide in a bitangent, their point of intersection is indeterminate. Hence the denominators in the above expressions must vanish, and we get $v-2=0$ and $u=0$, also $u=v=0$; but $u=v=0$ gives $t^2=0$, which gives an *undulation*, as has already been seen.

Thus $u=t+t'=0$, and $v=tt'=2$, whence $t^2+2=0$, and we get the bitangent $x+4y=0$.

Ex. 2. Find the singular points on the following curves :

(*i*) $x=t+t^{-1}$, $y=1+t^2$ $\qquad$ (*ii*) $x=t(2-t)$, $y=t^4(2-t)^3$

(*iii*) $x=1+t+t^2$, $\qquad y=t^3+t^2+t+1$.

Ex. 3. Find the singular points and singular lines on the curves :

(*i*) $x=1+t^2$, $y=t^4$, $z=t^2-t$

(*ii*) $x=\cos 3\phi$, $y=\sin 3\phi$, $z=\cos\phi$.

Ex. 4. Find the Plückerian characteristics of the curve $y^2z^3=x^5$

$[n=m=5,\ p=0,\ \delta=\tau=3,\ \kappa=\iota=3.]$

Ex. 5. Show that the envelope of lines joining corresponding points on a unicursal quartic and a conic is a class-sextic.

239. Special Class of Rational Curves:

Among the rational curves there are those with an $(n-1)$-*ple* point, or with three ordinary points where the tangent has n-pointic contact. To this class belong the curves called *Triangular Symmetric Curves*, represented by the equation $ax^n + by^n + cz^n = 0$, where n is rational, positive or negative.*

For a detailed study, see Darboux—*Comp. Rend.*, Vol. 94 (1890), pp. 930 and 1108, Brusotti—*Lomb. I. t. Rend.*, Vol. 37 (2), (1904), p. 888, Loria—Spezielle alg. und transzend. ebene Kurven, Vol. I (1910), p. 341, and Wieleitner—Spezielle ebene Kurven (1908).

240. The Circuit of a Unicursal Curve:

The co-ordinates of a point P on the curve being continuous functions of the variable parameter λ, the point P moves continuously as λ varies, provided, of course, P is finite. If the co-efficients in the equation of the curve

* Prof. Hilton has considered the case of the unicursal n-*ic* with three tangents of n-pointic contact. By a suitable choice of homogeneous co-ordinates, the equation of such a curve is thrown into the form

(*i*) $$x : y : z = t^n : (1-t)^n : -1$$

and $$x^{\frac{1}{n}} + y^{\frac{1}{n}} + z^{\frac{1}{n}} = 0, \quad \text{when } n \text{ is odd};$$

(*ii*) $$x : y : z = t^n : (1-t)^n : 1$$

and $$x^{\frac{1}{n}} \pm y^{\frac{1}{n}} \pm z^{\frac{1}{n}} = 0, \quad \text{when } n \text{ is even}.$$

In either case the curve can be projected to have the symmetry of the equilateral triangle. Hilton—*On plane curves of degree n with tangents of n-pointic contact*—Messenger of Mathematics, Vol. 49 (1920), p. 132. The general case has been discussed in Vol. 50 (1921) of the same Journal, pp. 31-40 and 171-76.

are all real, real values of λ will give only real points P. Consequently the real values of λ from 0 to 0 through infinity give a series of real points, ending where it began, *i.e.*, the points are arranged in a *single circuit*, which may pass through infinity, but the locus is still continuous ; therefore, a unicursal curve consists of a single circuit, with its real points arranged in one continuous series, *i.e.*, it is *unipartite*.

If, however, λ is imaginary of the form $\alpha+i\beta$, it is also of the form $\alpha-i\beta$, *i.e.*, the point P is a double point. Since a consecutive value of λ does not give a real point of the curve, there is no real point of the curve consecutive to P, *i.e.*, P is an acnode. Hence it follows that there may be a finite number of real intersections of imaginary branches, which are isolated points (acnodes), but these cannot be included in the continuous description of the curve by a real tracing point. Thus the unicursal curve consists of a single circuit, *i.e.*, it is unipartite.

241. Unipartite Curves not necessarily Unicursal :

Although unicursal curves are unipartite, the converse theorem is not always true, *i.e.*, all unipartite curves are not unicursal.

For example, consider the curve $x^3+x=y^2$, which consists of a single circuit, *i.e.*, is unipartite ; but it is not unicursal. If it is to be unicursal, the co-ordinates of its points must be expressible in terms of a single parameter, and the elimination of the parameter should give the equation of the curve. Since the co-efficients in the expressions may be either real or imaginary, by a proper substitution of the form $x=x'$, $y=\sqrt{i}y'$, $z=iz'$ the equation obtained is

$$x'^3-x'z'^2+y'^2z'=0$$

which is unicursal, but this is certainly bipartite, as can be easily verified, the branches lying between $x'-1$, x' and $x'+1$ and infinity. Hence the curve $x^3+x=y^2$, although unipartite, is not unicursal.*

242. Curves with Unit Deficiency:

It has been shown in § 225 that a curve with unit deficiency can be transformed into a cubic having the same deficiency, and as will be shown later on, the anharmonic ratio of the four tangents drawn from any point on a cubic is constant for all positions of the point. These facts enable us to express the co-ordinates of any point on a curve with unit deficiency as rational functions of a parameter θ, and of $\sqrt{\Theta}$, where Θ is a quartic function of θ. From what has been said above, it will be sufficient, if this can be established for a cubic curve.

For, in this case x, y, z can be expressed as rational functions of x', y', z'. If the cubic is taken to pass through the point xy, we may write $y=\theta x$ in its equation, when $x : y : z$ are obtained in the above form. The values of θ given by $\Theta=0$ correspond to the four tangents which can be drawn from the point xy to the curve.

Thus the co-ordinates of a point on the curve with unit deficiency can be expressed as rational functions of θ and $\sqrt{\Theta}$. Now by a linear transformation of θ, $\sqrt{\Theta}$ can be

* It is to be observed that the term *unicursal* makes no distinction between real and imaginary points, and if a unicursal curve has any real part, it consists of a single circuit; whereas the term *unipartite* refers to the appearance of the curve and takes cognizance of the real part only. In fact, just as an equation having a real root is not necessarily a simple equation, a unipartite curve is not necessarily unicursal.

brought to the form $\sqrt{(1-\theta^2)\,(1-k^2\theta^2)}$, and putting $\theta = \text{sn}u$, $\sqrt{(1-\theta^2)\,(1-k^2\theta^2)}$ becomes cn$u$ dn$u$ and the co-ordinates are expressible in terms of the elliptic functions sn$u$, cn$u$ and dn$u$ of a parameter u.

243. Co-ordinates in Terms of Elliptic Functions:

By using a method similar to that used in § 54, we shall now directly establish the theorem:

The co-ordinates of any point on a curve of unit deficiency can be expressed rationally in terms of elliptic functions.

Let $f=0$ be the equation of an n-*ic* with multiple points of orders k_1, k_2, k_3,...such that the deficiency

$$=\tfrac{1}{2}(n-1)(n-2)-\Sigma\tfrac{1}{2}k(k-1)=1$$

$$\therefore \quad \Sigma\tfrac{1}{2}k(k-1)=\tfrac{1}{2}(n-1)(n-2)-1=\tfrac{1}{2}n(n-3).$$

Hence, a unique curve of order $(n-3)$ can be drawn through the multiple points.

Now take a system of $(n-2)$-*ics* adjoint * to the given n-*ic*, *i.e.*, a system of $(n-2)$-*ics* having a $(k-1)$-*ple* point at each k-*ple* point of f, and passing through $(n-2)$ other fixed points on f.

But the number of arbitrary co-efficients in the equation of such an $(n-2)$-*ic* is $\frac{1}{2}(n-2)(n+1)$, and the number of conditions assigned is—

$$\Sigma\tfrac{1}{2}k(k-1)+(n-2), \quad i.e., \quad \tfrac{1}{2}(n-2)(n+1)-1$$

* We may take a pencil of N-*ics* subject to the condition that the intersections at the fixed points must be equal to $n\text{N}-2$, which is satisfied when the N-*ic* is an adjoint $(n-2)$-*ic*, passing through $n-2$ other fixed points, since $\Sigma k(k-1)+(n-2)=n(n-2)-2=n\text{N}-2$. A second condition is that the co-efficients must be connected by $\frac{1}{2}\text{N}(\text{N}+3)-1$ relations, which is again satisfied by the adjoint $(n-2)$-*ic*, since, $\frac{1}{2}\Sigma k(k-1)+(n-2)=\frac{1}{2}(n-2)(n+1)-1=\frac{1}{2}\text{N}(\text{N}+3)-1$.

Hence the equation of the system will contain one arbitrary parameter and can therefore be written as

$$u + \lambda v = 0 \qquad \ldots \quad (1)$$

where u and v are any two particular members of the system.

Now, the curves (1) intersect f, in general, in $n(n-2)$ points, of which $\Sigma k(k-1)+(n-2)$, *i.e.*, $n(n-2)-2$ are fixed.

There are then two variable points P and Q which depend on λ.

If we eliminate one of the variables y (say) between $f=0$ and $u+\lambda v=0$, we obtain an equation of order $n(n-2)$ in x, $n(n-2)-2$ of whose roots are the abscissæ of the fixed points and the remaining two are the abscissæ of the two variable intersections P and Q. Removing the known factors, we have an equation of the second degree in x with the co-efficients rational in λ, whose roots give the abscissæ of P and Q in the form—

$$x = \mathrm{A} \pm \mathrm{B}\Theta^{\frac{1}{2}}$$

where A and B are rational in λ, and Θ is a polynomial in λ. If we substitute either of these values in the equations of the n-*ic*, and the $(n-2)$-*ic*, we obtain two equations in y whose common root gives the ordinate of one of the points P (say) and is of the form $\mathrm{A}' + \mathrm{B}'\Theta^{\frac{1}{2}}$, where A′ and B′ are both rational in λ.

Since the values of λ given by $\Theta=0$ correspond to the points of contact * of the variable $(n-2)$-*ics* which touch $f=0$ at a point other than any of the fixed points, and

* These include cusps other than at the fixed points.

there are four such curves, Θ must be a polynomial of order four in λ.

Thus the co-ordinates of any point on a curve of unit deficiency may be expressed rationally in terms of a parameter λ and an expression of the form *

$$\Theta^{\frac{1}{2}} \equiv \{a\lambda^4 + 4b\lambda^3 + 6c\lambda^2 + 4d\lambda + e\}^{\frac{1}{2}}$$

By a suitable transformation, both λ and Θ can be simultaneously expressed as rational functions of the elliptic functions sn$u$, cn$u$, dn$u$, of u, which proves the proposition.

244. Simplification by Weierstrass's Notation:

Weierstrass's elliptic function $\wp(u)$ † is defined as—

$$\operatorname*{Lim}_{u=0} (u^2 \wp u) = 1$$

and is connected with its derived functions by the relation

$$\wp' u^2 = 4\wp^3 u - g_2 \wp u - g_3$$

where g_2 and g_3 are invariants.

The function $\wp u$ is *doubly periodic*. If ω and ω' are the periods, we obtain the same value of $\wp u$, when u is replaced by $u + m\omega + m'\omega'$, where m and m' are integers.

Now, we have seen that the co-ordinates of a point can be taken as $A + B\Theta^{\frac{1}{2}}$, where A and B are rational in λ, while Θ is of the form $(a\lambda^4 + 4b\lambda^3 + 6c\lambda^2 + 4d\lambda + e)$

In order to transform $\Theta^{\frac{1}{2}}$, we make the substitution—

$$\lambda = -\frac{b}{a} + \frac{1}{2}\frac{\wp' u - \wp' v}{\wp u - \wp v}$$

* Clebsch—Crelle, Bd. 64 (1865), p. 217.

† Goursat—Mathematical Analysis, Vol. II, Part I, § 69, p. 156.

where $\wp v$ and $\wp' v$ are defined by the relations—

$$a^2\wp v = -(ac-b^2), \quad a^3\wp' v = a^2d - 3abc + 2b^3$$

and the invariants g_2, g_3 are given by $a^2g_2 = ae - 4bd + 3c^2$

and $$a^3g_3 = ace + 2bcd - ad^2 - eb^2 - c^3.$$

Then $\Theta^{\frac{1}{2}}$ is transformed to $a^{\frac{1}{2}}[\wp(u) - \wp(u+v)]$, while A and B are transformed into rational functions of $\wp u$ and $\wp' u$.

Hence the co-ordinates $A \pm B\Theta^{\frac{1}{2}}$ are expressible rationally in terms of doubly periodic elliptic functions in the form—

$$\chi \pm \psi[\wp u - \wp(u+v)]$$

where χ and ψ are rational in $\wp u$ and $\wp' u$.

Ex. 1. Consider the cubic

$$y^2 = ax^3 + 3bx^2 + 3cx + d \quad \dots \quad (1)$$

Putting $y = 4y'/a$ and $x = -b/a + 4x'/a$,

the cubic is transformed into $y'^2 = 4x'^3 - g_2x' - g_3 \quad \dots \quad (2)$

where $16g_2 = 12(b^2 - ac)$ and $16g_3 = 3abc - 2b^3 - a^2d$.

But since $g_2^3 - 27g_3^2 \neq 0$, the equation (2) represents a non-singular cubic and is satisfied by $x' = \wp u$, $y' = \wp' u$, where g_2, g_3 are the invariants,

whence $$x = -\frac{b}{a} + \frac{4}{a}\wp u, \qquad y = \frac{4}{a}\wp' u.$$

Ex. 2. Express rationally in terms of elliptic functions the co-ordinates of any point on the curve $xz^2 = y(x-y)(k^2x-y)$.

The curve evidently passes through the point $x = y = 0$, which is a point of inflexion with $x = 0$ as the inflexional tangent.

The three lines $y = 0$, $x - y = 0$ and $y - k^2x = 0$ pass through this point and meet the curve elsewhere, where $xz^2 = 0$, *i.e.*, they are the tangents drawn from the point of inflexion to the curve, and their points of contact are situated on the line $z = 0$. This equation is identically satisfied, if we put $\rho x = t^3$, $\rho y = t$, $\rho z = \sqrt{1-t^2}\sqrt{1-k^2t^2}$.

Putting $t=\mathrm{sn}\,u$, $\sqrt{1-t^2}=\mathrm{cn}\,u$, and $\sqrt{1-k^2t^2}=\mathrm{dn}\,u$, and taking the radical with a determinate sign, we have—

$$\mathrm{sn}^2u+\mathrm{cn}^2u=1,\quad k^2\mathrm{sn}^2u+\mathrm{dn}^2u=1.$$

Thus $$\rho x=\mathrm{sn}^3u,\quad \rho y=\mathrm{sn}\,u,\quad \rho z=\mathrm{cn}\,u\ \mathrm{dn}\,u$$

We may consider the intersection of the curve with the line $x=y$, which gives $$x:y:z=\mathrm{sn}^3u:\mathrm{sn}\,u:\mathrm{cn}\,u\ \mathrm{dn}\,u.$$

Ex. 3. Show that the co-ordinates of any point on the quartic

$$y^2=ax^4+6cx^2+4dx+e$$

can be expressed in terms of Weierstrass's function $\wp u$ in the form—

$$x=\tfrac{1}{2}\frac{\wp'u-\wp'v}{\wp u-\wp v},\qquad y=\sqrt{a}\ \left[\wp u-\wp(u+v)\right]$$

where $$\wp v=-\frac{b}{a},\quad \wp'v=\frac{d}{a}.$$

Ex. 4. Show that the invariants g_2, g_3 in *Ex.* 3 are given by

$$a^2g_2=ae+3c^2,\qquad a^3g_3=ace-c^3-ad^2.$$

Ex. 5. Show that the sum of the arguments of the three points where any straight line meets a non-singular cubic is equal to the period.

Ex. 6. Express rationally in terms of elliptic functions the co-ordinates of any point on the curves :

$$(i)\ z^3(x^3+y^3)=x^3y^3\qquad (ii)\ y^3=(x-a)^2(x-b)^2$$

245. The Converse Theorem :*

If the co-ordinates of any point on a curve can be expressed rationally in terms of elliptic functions $\wp u$ and $\wp' u$, the curve is, in general, of unit deficiency.

Let the expressions for co-ordinates be—

$$x:y:z=f_1(u):f_2(u):f_3(u)$$

where f_1, f_2, f_3 are each of the form $A+B\wp'(u)$, A and B

* See H. Hilton—Plane Algebraic Curves, Chap. X, § 8.

being polynomials in $\wp u$. Since, by successive differentiation of the relation

$$[\wp' u]^2 = 4[\wp u]^3 - g_2 \wp u - g_3$$

we may express the products of powers of $\wp u$ and $\wp' u$ occurring in $A \pm B\wp' u$ linearly in terms of—

$$\wp u, \ \wp' u, \ \wp'' u, \ldots$$

f_1, f_2, f_3 may each of them be taken as a linear function of these in the form

$$a + b\wp(u) + c\wp'(u) + d\wp''(u) + \ldots + p\wp^{n-2}(u).$$

Any straight line $lx + my + nz = 0$ will meet this curve

where $$lf_1(u) + mf_2(u) + nf_3(u) = 0.$$

The left-hand side of the equation has *n poles*, and therefore, *n zeros*. Consequently, the curve is of order n.*

The equation of the tangent at any point of parameter u may be written as in §55, and the co-ordinates of the tangent are—

$$f_2 f'_3 - f_3 f'_2, \quad f_3 f'_1 - f_1 f'_3, \quad f_1 f'_2 - f_2 f'_1.$$

Now, in each of these expressions, the terms of the type

$$\wp^{n-2}(u)\wp^{n-1}(u)$$

cancel, and consequently, each, when reduced to linear form, can be written as—

$$\phi + a_0 \wp(u) + a_1 \wp'(u) + \ldots + a_{2n-2} \wp^{2n-2}(u).$$

$\therefore$ The class of the curve cannot be greater than $2n$.

Hence, as in § 55, the reduction in the class of the curve cannot be less than $n(n-1) - 2n$, or $n(n-3)$, and consequently, the number of double points (excluding

* Goursat—Math. Analysis, Vol. II, Part I, § 68.

cusps) cannot be less than $\frac{1}{2}n(n-3)$ or $\frac{1}{2}(n-1)(n-2)-1$, *i.e.*, the deficiency cannot be greater than 1.*

For a detailed discussion of the curves of unit deficiency, the reader is referred to the well-known paper of Clebsch—"*Ueber diejenigen Curven, deren co-ordinaten sich als elliptischen Functionen eines Parameters darstellen lassen*"—Crelle's Journal, Bd. 64 (1865), pp. 210-270. Also—Harnack—Math. Ann., Bd. 9 (1876), p. 1, and Porter—Trans. Am. Math. Soc., Vol. 2 (1901), p. 36.

* The deficiency is zero or unity. But since the functions f_1, f_2, f_3 will not usually be rational functions of a single parameter, we say that, in general, the deficiency is unity.

CHAPTER XII

THEORY OF CORRESPONDENCE

246. Correspondence of Points on a Curve:

In Chapter X, we have discussed the general principles of correspondence of points of two different planes, or of the same plane, as a whole. In this Chapter, we shall discuss the correspondence of points on the same curve, or on different curves.*

The simplest of such correspondences is illustrated by the homographic systems of lines and conics, the essentials of which are to be found in all treatises on conic sections.† In fact, there is a (1, 1) correspondence between the elements of two bases defined by the bilinear relation

$$A\lambda\lambda' + B\lambda + C\lambda + D = 0$$

where λ and λ' are the parameters of the corresponding elements.

When the two bases are superimposed, we obtain a correspondence between points of the same base, special cases of which are studied under the name *Involution* ranges or pencils.

Chasles extended and discussed the theory as applied to unicursal curves; but the theory is applicable to all curves generally, and we shall presently consider the general principles of correspondence of points on the same curve.

* Lefschetz, S., *Correspondences between Algebraic Curves*—Annals of Math. (2), Vol. 28, No. 3, pp. 342-54 (1927).

† Scott, Modern An. Geo., §§ 192-196.

Let P and P′ be two points on the same curve, such that to any position of P there correspond r' positions of P′, and to a given position of P′, r positions of P. Then the points of the curve are said to have an (r, r') correspondence. If $r=r'=1$, the correspondence is (1, 1), and rational.

The correspondence * between points of a curve may be instituted in various ways, as the following illustrations will show:

(1) The points of a conic may be put into a (1, 1) correspondence, if the points collinear with a given point in its plane is made to correspond, the points forming an involution range on the conic.

(2) Any radius vector through a fixed origin O meets a curve of order n in n points $P, P_1, P_2, P_3, \ldots P_{n-1}$. If P′ denotes any one of the points $P_1, P_2, \ldots P_{n-1}$, we may

* The principles of correspondence for points in a line was established by Chasles in his paper in the *Comptes rendus*, June-July, 1864. But prior to him De Jonquières considered the principle in 1860—"*L'œuvre mathematique d'Ernest de Jonquières.*" It has been extended to unicursal curves by Chasles in a paper of his—"*Sur les courbes planes ou à double courbure dont les points peuvent se dèterminer individuallement—Application du principle de correspondence dans la theorie de ces courbes*"—Comptes rendus, Vol. 62 (1866), p. 534. Cayley referred to the principle—Comptes rendus, Vol. 62 (1866), but gave a discussion in his memoir—*On the correspondence of two points on a curve* (Coll. Works, Vol. 6, p. 9), where he discussed only a particular case. Finally he gave a number of applications in his paper—*Second Memoir on the Curves which satisfy given conditions*—Coll. Works, Vol. 6, pp. 263-71. But an algebraic demonstration of a more general formula has been given by Brill—"*Ueber Entsprechen von Punktsystemen auf einer Curve*"—Math. Ann., Bd. 6 (1873), p. 33, and a geometrical treatment in his paper—"*Ueber die Correspondenzformel*"—Math. Ann., Bd. 7 (1874), p. 607. Lindemann has given a demonstration of the principle with the help of Abelian integrals—Crelle, Bd. 84 (1878), p. 301. See also a paper by G. Loria—Bibl. Math. (3) 3 (1902), p. 285.

say that to a given position of P there correspond $(n-1)$ positions of P′, and conversely, to any position of P′ there are $(n-1)$ positions of P. Hence there is an $(n-1, n-1)$ correspondence.

This will be a rational correspondence, if $n=2$ (Case 1), a particular case of which is afforded by the circular inversion §§ 15 and 217.

When the origin O lies on the curve, there will be an $(n-2, n-2)$ correspondence; and in general, if O is an r-ple point on the curve, there is an $(n-r-1, n-r-1)$ correspondence, *i.e.*, corresponding to any position of P there are $(n-r-1)$ positions of P′, and *vice versa*.

It is to be noted that a (1, 1) correspondence is possible for a cubic or a quartic with a node, but it is not always possible.

247. Non-symmetrical Correspondences:

In the preceding illustrations, the correspondence is symmetrical, *i.e.*, from either given point the other is obtained by the same construction. But there are correspondences which are not so symmetrical. The following illustrations will clearly explain what we mean:

(1) There may be instituted a (1, 2) correspondence between the points of a conic as follows:—

Let O be a point on the conic **S** and p a line in the same plane. A radius vector through O will meet the conic in a point P and the line in a point Q. The polar line of Q will meet **S** in two points P_1 and P_2. Thus there is a (1, 2) correspondence between P and P_1, P_2 (P′).

(2) The tangent at any point P of an n-*ic* meets the curve in $(n-2)$ other points, so that to any position of P there correspond $n-2$ positions of P′. But if P′ is given, P may be any one of the points of contact of the $(m-2)$ tangents which can be drawn from P′ to the

curve, m being the class. Thus to any given position of P′, there are $m-2$ positions of P, and there is consequently an $(m-2,\ n-2)$ correspondence.

248. Analytical Discussion:

The preceding examples show that a geometrical construction can be given for determining a correspondence on a given curve $f=0$.

An algebraic correspondence (l, k) between the points P, P′ of two curves, or of the same curve, is defined by a system of algebraic equations between the co-ordinates of P and P′, such that when P is given, there are k points P′, and when P′ is given, there are l points P. In fact the corresponding points are obtained as the intersections of a certain curve Θ with $f=0$. In the above illustrations this curve Θ was taken to be a right line. It may again happen that some of the intersections P′ coincide with the given point P, and these points must then be excluded.

Let the equation of the curve Θ be given in the form

$$\Theta\ [(x,\ y,\ z)^l\ ;\ (x',\ y',\ z')^k]=0$$

which contains (x, y, z) in degree l and (x', y', z') in degree k.

If P (x, y, z) be given, this equation represents a curve Θ_k of order k in (x', y', z'); and if P′(x', y', z') be given, a curve Θ_l of order l in (x, y, z). The two curves Θ_l and Θ_k intersect the given curve $f=0$ in ln and kn points respectively, and therefore we obtain a $(ln,\ kn)$ correspondence. If, however, the curve Θ_l for a given point (x', y', z') passes through the same, that point is to be excluded, so that if α intersections of Θ_l with f coincide at P′ (x', y', z'), the number of remaining intersections P is $ln-\alpha$. Similarly, if the curve Θ_k for any given point (x, y, z) meets $f=0$ in β points coinciding with (x, y, z), there are $kn-\beta$ remaining intersections P′. Thus we

obtain a $(ln-\alpha,\ kn-\beta)$ correspondence, and we denote it by $(ln-\alpha,\ kn-\beta)$.

Ex. In *Ex.* 2, § 246, we have $l=k=1$, also $\alpha=\beta=1$, and the correspondence is $(n-1,\ n-1)$.

In *Ex.* 2, § 247 $ln=m$, $k=1$, $\alpha=\beta=2$, and the correspondence is $(m-2,\ n-2)$.

249. United Points:

As in the case of general correspondence, we have elements coinciding with its correspondents, so in this particular case, a point may correspond to itself, and is then called a *united point*. For example, in the case of involution range of points on a conic (*Ex.* 1, § 246), the points of contact of the tangents drawn from the fixed point O to the conic are united points. In general, if the corresponding points are collinear with a fixed point, the united points are the points of contact of the tangents drawn from the fixed point to the curve. Hence the number of such points is equal to the class of the curve.

Prof. Cayley * explained by means of a number of illustrations the general formula for finding the united points without any formal proof. But later on Brill † gave a formal accurate proof of the formula.

In the equation of § 248, if we put $x=x'$, $y=y'$, $z=z'$, we obtain an equation $\Theta_{l+k}=0$, of order $l+k$, which now represents a curve of order $l+k$ passing through all those points for which Θ_k passes through (x, y, z), and Θ_l passes through (x', y', z'). Thus the curve $\Theta_{l+k}=0$ intersects

* Cayley—" Note sur la correspondence, etc." Comp. Rend. Ac. Sc., Paris (1866), Vol. 62, pp. 586-590, also "Correspondence of two points on a curve"—Coll. Works, Vol. 6, pp. 9-13.

† Brill—Ueber Entsprechen von Punktsytemen auf einer Curve—Math. Ann. Bd. 6 (1873), pp. 33-65, and " Ueber die correspondenzformel,"—Math. Ann. Bd. 7 (1874), pp. 607-622.

the curve $f=0$ at the *united* points. Therefore, the correspondence $(ln, kn) \equiv (a, b)$ has $(l+k)n = ln + kn \equiv a+b$ united points as given by Chasles.

If, however, Θ_k meets f in one or more points (α) coincident at (x, y, z), and Θ_l meets f in one or more points (β) coincident at (x', y', z'), which, of course, happens, when they are singular points on f, these α and β points are not to be included in the number of united points, and the order of multiplicity in α and β diminishes the number of such points.

In the preceding examples, $\alpha=\beta=1$, and the formula holds for rational curves. In fact, when $\alpha=\beta$, the investigation becomes much simplified and the correspondence depends upon the value of α, and is then denoted by the symbol $(ln-\alpha, kn-\alpha)_\alpha$.

In the example (1) of § 247, there are three united points, namely, the points M_1, M_2, where p cuts the conic and the point O.

Again, P_1 may coincide with O, while P_2 is distinct, or both P_1, P_2 may coincide with M_1, M_2; but this does not stipulate any higher species of united points, as has been shown by Cayley in the general case.*

250. Chasles' Correspondence Theorem :

Let C_n and C_n' be two curves of orders n and n' respectively, such that there is a (1, 1) correspondence between them, *i.e.*, to each point P of C_n there corresponds one, and only one, point P′ of C_n', and *vice versa*. Let us first determine the class of the envelope of PP′, or in other words, let us see how many of the lines, such as PP′, pass through any point (0, 0, 1) for example.

* Cayley—Second memoir on curves which satisfy given conditions—the principles of correspondence—Coll. Works, Vol. 6 (1868), p. 265.

Consider a line p through the vertex O (0, 0, 1) which intersects C_n in n points, corresponding to which there are n points on C_n'. The n lines joining O to these n points on C_n' are given by an equation of the form—

$$\phi_n \equiv a_0 x^n + a_1 x^{n-1} y + \ldots + a_n y^n = 0$$

Any other line q through O will similarly correspond to n lines determined by the equation—

$$\phi'_n \equiv a'_0 x^n + a'_1 x^{n-1} y + \ldots + a'_n y^n = 0.$$

The pencil of lines $p + \lambda q = 0$, which, for simplicity, may be taken identical with $x + \lambda y = 0$, then determines an involution pencil of order n,

$$\phi_n + \lambda \phi'_n = 0 \qquad \ldots \quad (1)$$

Similarly, considering the intersections of the lines p and q with C_n', we obtain an involution pencil of order n',

$$\psi_{n'} + \lambda \psi'_{n'} = 0 \qquad \ldots \quad (2)$$

where $\psi_{n'}$ and $\psi'_{n'}$ are binary expressions of order n', similar to ϕ_n and ϕ'_n.

The two involutions (1) and (2) are then projective and have the same vertex. The double or self-corresponding elements are obtained by eliminating λ between (1) and (2) in the form

$$\phi_n \psi'_{n'} = \phi'_n \psi_{n'} \qquad \ldots \quad (3)$$

The equation (3) is of order $n + n'$, and therefore gives $n + n'$ self-corresponding rays of the two pencils, whence the class of the envelope of PP′ is determined. (§ 152.)

The above considerations hold for two ranges of points as well, and we obtain Chasles' Correspondence Theorem :

If there are two superimposed systems of elements, such that to each element of the first system correspond n

elements of the second, and to each element of the second, n' elements of the first, then there are $n+n'$ self-corresponding elements; or in other words:

In an (n, n') correspondence on the same base, there are $n+n'$ double elements, as we have otherwise determined in the preceding article.

251. Correspondence Index or Characteristic:

Let A and A′ be any two points in the plane of the n-*ic* $f=0$, and draw a line p joining A to any point P (x, y, z) on $f=0$. Determine the points P′ (x', y', z') on f which correspond to P, and to all other points where the line p meets the n-*ic*, and join these points to A′ by means of lines p'. Then the locus of intersection of the corresponding lines p and p' is a curve Θ, which can as well be obtained if we start with a point P′ $(x', y' z')$. This curve Θ then meets the curve $f=0$ at the *united* points. If α united points coincide at each point P (x, y, z), Θ must pass through these points, and must contain f as part α times. Therefore, to each line p' the line p corresponds α times, and these α-*ple* line p passes through P′(x', y', z'). Hence we have $\alpha=\beta$ (§ 248).

But since A and A′ may be any points in the plane, we require only to determine how many of the points (x', y', z'), common to Θ_k and f, coincide at (x, y, z). The number α of these points is called the "*index*" of the correspondence which is denoted by—

$$(ln-\alpha,\ kn-\alpha)_\alpha, \quad \text{or,} \quad (a-\alpha,\ b-\alpha)_\alpha.$$

252. Common Elements of Two Correspondences:

Let us consider on the curve $f=0$ the two correspondences

$$\phi \equiv (a, b)_\circ \quad \text{and} \quad \phi' \equiv (a', b')_\circ$$

in both of which the correspondence index or *characteristic* is zero. We shall now determine the pairs of points (x, y, z), (x', y', z') on f, which simultaneously satisfy the two correspondences, *i.e.*, we shall find the number of points of the two correspondences which correspond to the same point (x_1, y_1, z_1) on f. This number is determined by the intersections of f with a certain curve ψ.

To determine the order of the curve ψ, we count the number of points in which any line L will intersect it. In virtue of the second correspondence ϕ', to any point (x, y, z) of this line there correspond b' points (x_1, y_1, z_1) on f, which are obtained by its intersection with a curve Θ'_k. To each of these b' points correspond, in virtue of the first correspondence ϕ, $b'l$ points (x', y', z') on L given by the curve Θ_l. To each of these points again correspond in the same way $a'k$ points (x, y, z) on L. Consequently, on the line L we have a $(b'l, a'k)$ correspondence of points (x, y, z) and (x', y', z'). The order of the curve ψ, which may be considered as generated by two pencils of lines, is consequently $(b'l + a'k)$.

Hence, ψ intersects $f=0$ in $n(b'l + a'k) = ab' + a'b$ common points (x, y, z) and (x', y', z'), or, $ab' + a'b$ pairs of points (x, y, z), (x', y', z'), which satisfy both the correspondences. The same result may be obtained in a different manner as follows:

To each point (x', y', z') of the plane correspond the ll' points of intersection (x, y, z) of the curves $\Theta_l = 0$ and $\Theta'_{l'} = 0$ belonging to the two correspondences. Similarly, to each point (x, y, z) there are kk' points (x', y', z'). If (x', y', z') moves along a line, the ll' correspondents (x, y, z) move along a curve of order $(k'l + kl')$. Consequently, if (x', y', z') describes the curve f, (x, y, z) describes a curve of order $n(k'l + kl')$.

Each point of intersection (x, y, z) of this curve with f together with a point (x', y', z') on f will give the required

pair of points. The number of such pairs satisfying both the correspondences (a, b), (a', b') on f is, therefore, equal to

$$n^2(k'l+kl')=(nk.nl'+nk'.nl)=a'b+ab'.$$

If, however, the points (x, y, z) and (x', y', z') are symmetrical with regard to both the correspondences, *i.e.*, if $a=b$, $a'=b'$, we shall, as before, obtain the same curve of order $n(kl'+k'l)$. Each point of intersection with f will be taken twice to give a couple. The number of pairs therefore will be equal to half the number, *i.e.*, $l=aa'=bb'$.

Ex. Let $P=0$ and $Q=0$ be the equations of two points. Then

$$P+\lambda Q=0 \quad \text{and} \quad P+\mu Q=0$$

represent two points on the line joining P and Q. If there is a $(1, 1)$ correspondence between the points, it is expressed by

$$\Theta(\lambda, \mu)=a\lambda\mu+b\lambda+c\mu+d=0 \tag{1}$$

We may have a second $(1, 1)$ correspondence on the same line given by

$$\Theta'(\lambda, \mu')=a'\lambda\mu'+b'\lambda+c'\mu'+d'=0 \tag{2}$$

These two correspondences Θ and Θ' must have then, by the theorem, $1.1+1.1=2$ common pairs. The same value of λ will make $\mu=\mu'$, if

$$\begin{vmatrix} a\lambda+c & b\lambda+d \\ a'\lambda+c' & b'\lambda+d' \end{vmatrix}=0$$

This is a quadratic in λ and therefore gives two values λ_1 and λ_2 of λ for which $\mu=\mu'$. If λ is eliminated between Θ and Θ', we obtain a bilinear relation between μ and μ', of the form :

$$\begin{vmatrix} a\mu+b & c\mu+a \\ a'\mu'+b' & c'\mu'+a' \end{vmatrix}=0$$

Hence, μ, μ' determine a $(1, 1)$ correspondence. If we put $\mu=\mu'$, we obtain a quadratic equation giving the united points of this correspondence. It easily follows then that the common pairs are given by

$$\lambda_1/\mu_1, \quad \lambda_2/\mu_2.$$

253. If one or both the correspondences have a point where one or more corresponding points coincide, then to determine the common points (x, y, z), (x', y', z') of the correspondences, the number $ab'+a'b$ must be reduced.

Consider the correspondences—

$$\phi \equiv (a, b)_0 \quad \text{and} \quad \phi' \equiv (a'-\gamma', b'-\gamma')_{\gamma'}.$$

In this case the formula of the preceding article holds, if only distinct pairs of points are taken into account. Hence the number must be reduced by the number of coincident corresponding points, but such coincidence takes place γ'-times only at the united points of ϕ at each of which ϕ' has always a γ'-point, and consequently it is equivalent to $\gamma'(a+b)$.

The number of distinct pairs of points which satisfy simultaneously the two correspondences is then equal to

$$ab'+a'b-\gamma'(a+b)=a(b'-\gamma')+b(a'-\gamma').$$

254. Common Pairs:

We shall now consider the two correspondences of indices γ and γ' respectively, *i.e.*,

$$\phi \equiv (a-\gamma, b-\gamma)_\gamma, \quad \phi' \equiv (a'-\gamma', b'-\gamma')_{\gamma'}.$$

Let us deform the correspondence ϕ' a little into a new correspondence $\phi_1 \equiv (a', b')_0$, so that the present case is reduced to the preceding one. The correspondence ϕ_1 has no γ'-point at $(x)=(x')$ but gives, on the other hand, on f a series of γ' points contiguous to all points x' or x. The number of pairs of points common to the two correspondences ϕ_1 and ϕ is, by the preceding article, equal to

$$ab'+a'b-\gamma(a'+b').$$

Among these pairs, there are those consisting of two still

more contiguous points x, x', which will coincide, and indicate a further reduction, when ϕ_1 is again deformed.

Let the point (x') move on f, then the γ' points (x) of ϕ_1 become contiguous to it. But if (x') moves up to coincidence with a similar united point (x_1) of ϕ, then also the series of points on ϕ_1 moves up to coincidence with (x_1), and finally coincides with it, and then moves away, as (x') proceeds further.

Hence we conclude that the common pairs of points of ϕ and ϕ' will be obtained by deforming ϕ_1 back to ϕ', each of the N united points of ϕ will occur γ' times.

Therefore the number $\mathcal{N}$ of all the distinct common pairs of ϕ and ϕ' is given by—

$$\mathcal{N} = ab' + a'b - \gamma(a' + b') - \gamma' \mathrm{N}$$

Similarly, by deforming ϕ to ϕ_1, we obtain—

$$\mathcal{N} = a'b + ab' - \gamma'(a + b) - \gamma \mathrm{N}'$$

where N′ denotes the number of united points of the correspondence ϕ'.

By identifying these two expressions, we obtain

$$\frac{\mathrm{N} - (a - \gamma) - (b - \gamma)}{\gamma} = \frac{\mathrm{N}' - (a' - \gamma') - (b' - \gamma')}{\gamma'} \equiv \mathrm{M} \quad \text{(say)}$$

which is independent of the intermediate correspondence. Therefore M depends on f, and in fact, on its deficiency and is to be determined by considering a special case.

255. Cayley-Brill's Correspondence Formula:

Let us take the correspondence between the point of contact (x') of a tangent to the curve $f=0$ and the $(n-2)$ other points (x) where the tangent meets f. In this case

the number of united points is evidently equal to the number of points of inflexion of f, *i.e.*, $3n(n-2)$

and $$a=n, \quad b=n(n-1), \quad \gamma=2$$

$$\therefore \mathrm{M}=\frac{3n(n-2)-(n-2)-\{n(n-1)-2\}}{2}=(n-1)(n-2)=2p$$

where p is the deficiency.

Hence we obtain the correspondence formula for united points of ϕ: $\quad \mathrm{N}=(a-\gamma)+(b-\gamma)+2\gamma p.$*

Thus, for the correspondence $\phi\equiv(\alpha, \beta)_\gamma$ the number of united points on a curve f of deficiency p (assuming there are only double points, etc.) is given by—

$$\mathrm{N}\equiv\alpha+\beta+2\gamma p.\dagger$$

This is known as Cayley-Brill's *Correspondence Formula*, and it is easily seen that Chasles' Formula holds when either γ or p or both are zero.

If we substitute this value of N in one of the formulae for $\boldsymbol{N}$, we obtain for the common united pairs $\boldsymbol{N}$ of the two correspondences—

$$\phi\equiv(a-\gamma,\ b-\gamma)_\gamma, \qquad \phi'\equiv(a'-\gamma',\ b'-\gamma')_{\gamma'}$$

$$\begin{aligned}\boldsymbol{N}&=ab'+a'b-\gamma(a'+b')-\gamma'\mathrm{N}\\ &=ab'+a'b-\gamma(a'+b')-\gamma'\{(a-\gamma)+(b-\gamma)+2\gamma p\}\\ &=(a-\gamma)(b'-\gamma')+(b-\gamma)(a'-\gamma')-2p\gamma\gamma'\\ &=\alpha\beta'+\alpha'\beta-2\gamma\gamma'p.\end{aligned}$$

* For a complete discussion of the united points, etc., the student is referred to Clebsch—Leçons sur la Geometrie, Vol. II, Chap. 1, pp. 146-188.

† This formula was first given by Cayley—*Comp. Rend.*, Vol. 62 (1866), p. 586, and Proc. Lond. Math. Soc., Vol. I (1866), p. 1, and was later on proved by Brill—Math. Ann. Bd. 6(1873), p. 33 ; Bd. 7(1874), p. 607. Several other proofs, etc., were given by Bobek, Segre, Lindemann, Zeuthen, etc. See also Severi—Torino Mem., Vol. 50 (2), (1901), p. 82, Vol. 54(2), (1903), p. 1, and Torino Atti, Vol. 38 (1903), p. 158.

where $\alpha = a - \gamma$, $\beta = b - \gamma$, $\alpha' = a' - \gamma'$, $\beta' = b' - \gamma'$.

Hence the common united pairs of the two correspondences $\phi \equiv (\alpha, \beta)_\gamma$ and $\phi' \equiv (\alpha', \beta')_{\gamma'}$ is $\alpha\beta' + \alpha'\beta - 2\gamma\gamma' p$.

The case of any number of algebraic correspondences on an algebraic curve was discussed by Hurwitz in *Math. Ann.*, Bd. 28 (1887), p. 561. The same was discussed and results geometrically interpreted by C. Rosati—*Sulle correspondence algebriche fra i punti di una curva algebrica*—Rendiconti della R. Accademia dei Lincei, Vol. 22 (1913), and Annali di matematica, Vol. 25 (3), (1916), pp. 1-32.

256. Applications of the Formula:

(1) *The Class of a Curve.*

Let the corresponding points be collinear with a fixed point O. Then the united points are the points of contact of the tangents through O, *i.e.*, the number of united points is equal to the class of the curve. There is an $(n-1, n-1)$ correspondence, and $\gamma = 1$. Here the cusps also occur as united points. For, it seems that the line joining O to a node or a cusp meets the curve at two points there, but in the caseJo a node the two points belong to different branches and are not consecutive, while at a cusp they are consecutive. Hence the line drawn through a node is *not* a tangent, while that through a cusp is a tangent only in a restricted sense. Thus, the cusps, in a restricted sense, and *not* the nodes, are united points. If there are κ cusps and m denotes the class, the united points are $m + \kappa$ in number, and by the present theorem, we have

$$\mathrm{N} = m + \kappa = 2(n-1) + 2p$$

$$\therefore \quad m = 2(n-1) - 2p - \kappa$$

$$= n(n-1) - 2\delta - 3\kappa \ (\S\ 121).$$

In the case of a unicursal curve, we have—

$$N = m + \kappa = 2(n-1). \qquad \therefore \quad m = 2(n-1) - \kappa.$$

(2) *Inflexions.*

Let P correspond to its tangential point P′ (see Ex. 2, § 247). Here the proper united points are the inflexions, and cusps are such in a special sense. There is an $(m-2, n-2)$ correspondence, so that

$$\alpha = m-2, \quad \beta = n-2 \quad \text{and } \gamma = 2,$$

since the line Θ meets f in two points at P.

If there are ι inflexions and κ cusps,

$$\iota + \kappa = (m-2) + (n-2) + 2.2.p = (m+n-4) + 4p.$$

Putting $2p = m - 2n + 2 + \kappa$, we get $\iota = 3(m-n) + \kappa$ (§149).

In the case of a unicursal curve, we have the number of united points equal to $N = \iota + \kappa = m + n - 4$

$$\therefore \quad \iota = m + n - 4 - \kappa.$$

Ex. A conic may have five-pointic contact at any point P of a cubic. This conic therefore meets the cubic in another point P′. Between P and P′ there is then a $(1, \omega)_5$ correspondence, where ω is the number of conics drawn through P′ having five-pointic contact elsewhere.

The united points of this correspondence will be those where a conic has six-pointic contact. These are called "*sextactic*" points.

Now, in the case of a non-singular cubic curve, the number of sextactic points, as we shall show later on, is 27. Thus with the help of the correspondence formula we may find the value of ω. Here $p = 1$.

$$\therefore \quad 27 = 1 + \omega + 2 \cdot 5.1 \qquad \text{or} \quad \omega = 16.$$

We thus obtain the theorem :

Through any point of a non-singular cubic sixteen conics can be drawn which will have five-pointic contact with the cubic elsewhere.

CHAPTER XIII

Higher Singularities on Curves

257. Historical Notes:

The theory of singularities on plane curves was first studied by Plücker in his great work the *Theorie der Algebraischen Curven* (1839), in which he considered some of the higher singularities. But the importance of the analysis of higher singularities has been recognised from the time of Cramer.* The subject has been studied by Cayley,† Halphen,‡ H. J. Smith,§ Brill and Noether.‖ Finally, Scott,¶ in her well-known paper, gave a number of highly interesting geometrical methods of dissolving higher singular points on curves by means of a number of simple illustrations. Other workers on the subject are Bertini, Zeuthen Segre, Cremona, etc.

In Chapter VII we have discussed the six equations of Plücker with regard to the nodes, cusps, etc., which are termed "ordinary singularities" of curves. But there are other kinds of singular points of a complex nature which are called "higher singularities," as distinguished from the ordinary singularities.

* Cramer—Introduction à l'analyse des lignes Courbes, Genève (1705).

† Cayley—On the Higher Singularities of a Plane Curve—Quarterly Journal of Mathematics, Vol. VII (1866), pp. 212-22.

‡ Halphen—Comptes Rendus, Vol. 78 (1874), p. 1105, and Vol. 80 (1875), p. 638.

§ H. J. Smith—On the Higher Singularities of Plane Curves—Proc. of the London Math. Soc., Vol. VI (1873-74), p. 153.

‖ Brill & Noether—A number of papers published in the Mathematischen Annalen, Bd. IX (1876), XVI (1880), XXII (1884).

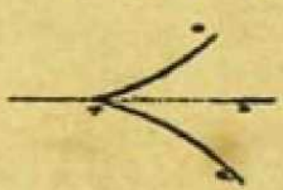
¶ Scott—On the Higher Singularities of Plane Curves—American Journal of Mathematics, Vol. XIV (1892), pp. 301-25.

In this Chapter we shall discuss the nature of different species of singularities and their effects on Plücker's numbers.

258. Species of Cusps:

We have seen in § 48 that the cuspidal tangent touches at the cusp both the branches which, however, may lie on the same side or opposite sides of the tangent. Cusps are accordingly divided into two species, as shown in the accompanying figures.

Case I: The two branches AP, AQ of the curve touch the tangent AT at A and lie on opposite sides of it. A is then called a cusp of *the first species*, or, a *keratoid* cusp (*i.e.*, cusp like a *horn*).



Case II: The two branches touch the tangent AT on the same side. This is called a cusp of *the second species*, or a *Ramphoid cusp* (*i.e.*, cusp like a *beak*).

Ex. Consider the nature of the origin on the curve $(y-x^2)^2=x^5$.

Any positive values of x give real values of y. Writing the equation in the form $y=x^2 \pm x^{\frac{5}{2}}$, it is seen that the values of y will be positive for small values of x, whether the upper or lower sign is taken, since the second term is less than the preceding when x is small.

The x-axis is a tangent and the two branches lie on the upper side. The origin is therefore a *ramphoid* cusp.

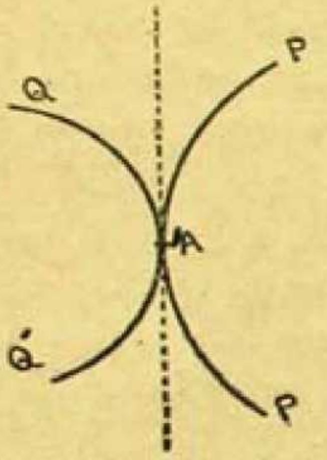
The analytical triangle gives the approximate form near the origin as that of the curve $(y-x^2)^2=0$, which represents two coincident parabolas for the two branches. For a second approximation another term is taken into account and the branches are given by—

$$y=x^2 \pm x^{\frac{5}{2}}$$

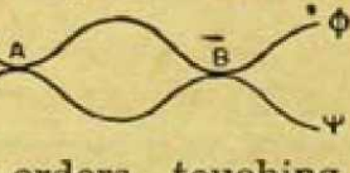
The axis of x touches both the branches and, in fact, has four-pointic contact with the curve. It is regarded as equivalent to four tangents that can be drawn from the origin to the curve.

259. Double Cusps:

It may also happen that the two branches of the curve, instead of extending towards one extremity of the tangent, extend towards both extremities, as shown in the adjoining diagrams. In these cases, there is a *double cusp*, which is

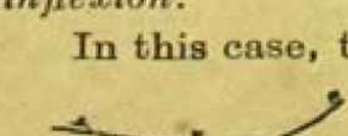


formed by the two branches of a curve touching at the point. Prof. Cayley calls it a *tacnode*. This point is indeed a distinct singularity, different in nature from ordinary singularities; because the tangent at such a point has in fact four points along the curve, namely, two points on each branch. These may arise when a curve F consists of two curves ϕ and ψ of lower orders, touching each other at a number of points A, B, etc.



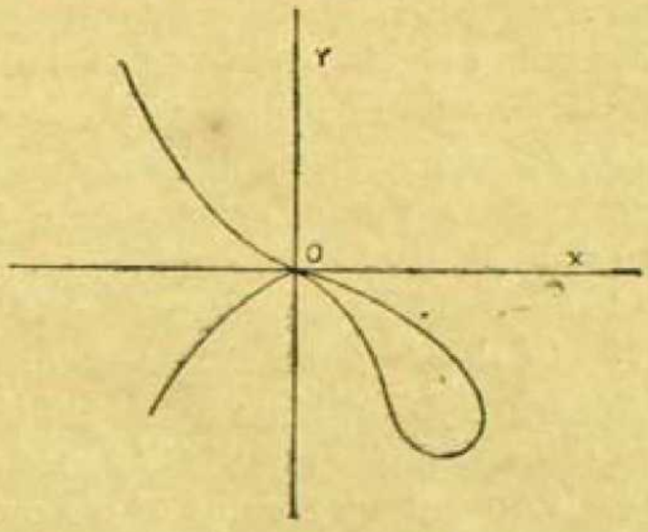
It may further happen that a double-cusp is of the first species towards one extremity and of the second species on the opposite extremity of the tangent. When the cusp is of different species towards opposite extremities of the tangent, Cramer calls the point *a point of oscul-inflexion*.

In this case, the point is a point of inflexion on one branch of the curve. It is evident then that all these properly belong to the class of higher singularities which we proceed to consider presently.



Ex. Consider the curve $y^2 + 2x^3y + x^7 = 0.$

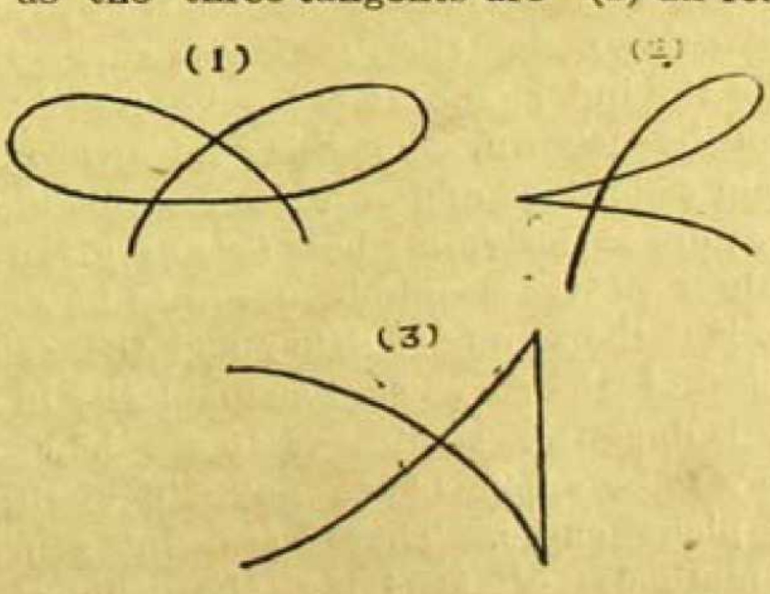
The curve is not symmetrical about either axis. It passes through the origin, but does not cut the axis again.



Evidently, there is a cusp at the origin with $y = 0$ as the cuspidal tangent; and in fact, there is a double-cusp at the origin, which is of the first species on the negative side of the y-axis and of the second species on the positive side. The point is an oscul-inflexion, and its shape is shown in the diagram.

260. Classification of Triple Points:

Triple points are classified into two main divisions according as the three tangents are—(*a*) all real, (*b*) one



real and two imaginary. The class of three real tangents

may again be subdivided into three species, according as the tangents are (1) all three distinct, (2) two coincident, (3) all three coincident. Thus there are in all *four* species of triple points. But a triple point may be regarded as arising from the union of three double points. The accompanying figures show the penultimate forms of these points and how these double points are about to unite to form the triple point in the different cases. It is formed by the union of three crunodes in the first case, two crunodes and a cusp in the second, and a crunode and two cusps in the third case. In this last case, however, the point does not visibly differ from an ordinary point on the curve.

In the case (*b*), the triple point is formed by the union of an acnode with an ordinary point of a curve, *i.e.*, a real branch of the curve passes through an acnode, and the singular point does not appear to differ from any ordinary point on the curve.

261. Equivalent Singularities:

Prof. Cayley has shown that any higher singularity whatever may be regarded as equivalent, in a perfectly definite manner, to a certain number of the simple singularities—the node, the ordinary cusp, the double tangent, and the inflexion. It must be noted, however, as Halphen has shown, that this equivalence is possible under certain conditions and within certain limits. We therefore require to determine how for any given singularity the values of these numbers are to be ascertained, so as to produce the same deficiency and the same effect on the class of the curve as the singularity in question. When this is done, we shall have to see how Plücker's equations are to be applicable to any singularities whatever of a plane curve. There are, in general, two principal methods of studying the subject—(1) by uccessive quadric transformations,—(2) by expansions.

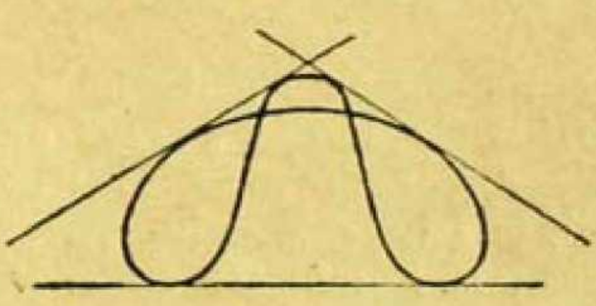
We shall in this Chapter explain the essentials of both these processes, one after the other; but it is convenient at the outset to exhibit the effect of such singularities by means of a simple illustration.

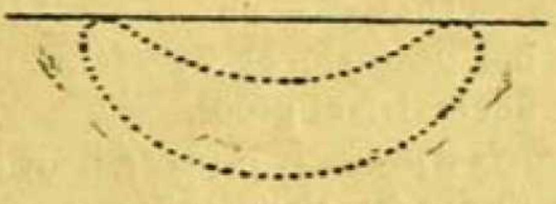
Ex. Consider the curve $x^4+y^4-4x^2y+y^2=0$.

The curve belongs to the class represented in § 190 and possesses a tacnode at the origin. The class of the curve can be determined by counting the number of tangents passing through any point, the point at infinity on the axis of x, for example. If the tangent at the origin is regarded as a tangent to the two branches, and each proper bitangent is counted as two, the class is found to be 8. But the class of a curve of order 4 is, in general, 12.

This diminution of 4 tangents is due to the presence of a higher singularity at the origin, and the effect is the same as due to the presence of two nodes, which unite to give rise to the tacnode, assuming, of course, there is no other singular point.

Two proper bitangents coincide with the tangent at the origin,





and there are six other bitangents. Hence the origin is not an ordinary singularity, nor the x-axis an ordinary singular tangent. In fact, the origin is a singular point both in point and line singularities.

That the class is reduced by 4 can also be seen from the fact that the first polar of any point meets both the branches in two points coinciding with the origin.

262. Analysis of Higher Singularities:

The principles of the theory of analysis of higher singularities of a plane curve are contained in the following theorem:

Every irreducible algebraic curve can be transformed by a birational transformation into one having no singularities except double points with distinct tangents.

Several proofs * of this fundamental theorem have been given by different mathematicians since the time of Kronecker who was the first to state the theorem, although in a slightly different form. Prof. Bliss of Chicago in a paper comments on the different proofs, and holds that the proofs given by Hensel-Landsberg and Walker are to be regarded as the best, although both of them are complicated and lengthy. Bliss in another paper, † however, gives a proof which he claims to be simpler than any of its predecessors and is an extension of the method of Kronecker.

Prior to all this, Nöther ‡ gave the following theorem :

Each irreducible algebraic plane curve can be transformed into another which possesses only ordinary singularities, i.e., multiple points with distinct tangents, by a series of quadric transformations of the plane, i.e., by Cremona transformations.

That the number of these transformations is finite has been shown by Hamburger, § while Bertini ‖ gave a geometrical proof.

* Kronecker—Crelle, Bd. 91 (1881), p. 301. Hensel-Landsberg—Theorie der algebraischen Functionen, § 2, p. 402. Also Hensel—Encyclopädie der mathematischen Wissenschaften II. c. 5, § 25. Halphen—Comptes Rendus, Vol. 80 (1875), p. 638, and also J. de Math. Bd. 2 (3), (1876), p. 87. Bertini—Rivista di Matematica, Vol. I (1891), p. 22, or Math. Ann. Bd. 44 (1894), p. 158. Walker—On the resolution of higher singularities of algebraic curves into ordinary nodes—Dissertation, Chicago (1906). G. A. Bliss—The reduction of singularities of plane curves by birational transformation—Bull. of the Am. Math. Soc., Vol. 29 (1923), pp. 161-183.

† G. A. Bliss—"Birational transformation simplifying singularities of algebraic curves"—Trans. of the Am. Math. Soc., Vol. 24 (1922).

‡ Nöther—Göttingen Nachrichten (1871), p. 267, also Math. Ann. Bd. 9 (1876), p. 166, and Bd. 23 (1884), p. 311.

§ Hamburger—Zeitschrift für Mathematik und Physik—Bd. 16 (1871), p. 461.

‖ Bertini—Reale Istituto Lombardo di Scienze e Lettere—Rendiconti (Millano), Vol. 21 (2), (1888), pp. 326, 413.

263. Successive Transformations:

In order to resolve a *k-ple* point A on the curve $f=0$, we take A and any two other points B and C as the vertices of the fundamental triangle of a quadric transformation. Then the transform f' has at the fundamental points A′, B′, C′ of the transformed plane ordinary multiple points, while the singularities of f, other than A, are transformed into multiple points of f' of the same order and form as the original (§ 214). If some of the k-tangents at A coincide in the lines $t_1, t_2, t_3, \ldots$, then there is on the line B′C′ an equal number of points $A'_1, A'_2, \ldots$ (other than B′ or C′) which are multiple points on f' of orders (say) $k_1, k_2, k_3, \ldots$, such that $k_1+k_2+k_3+\ldots \leq k$. These points, however, can all of them unite to form a single *k-ple* point A′.

Again, we apply on f' another quadric transformation with one fundamental point at A′ and so on. By a finite number of such operations, we may obtain from A multiple points of lower orders ; and finally a curve ϕ will be obtained on which the points corresponding to A are all ordinary points. The ordinary multiple points of orders $k_1, k_2, k_3, \ldots$ indefinitely adjacent to the *k-ple* point A on the tangents $t_1, t_2, t_3, \ldots$ are said to form the "*neighbourhood*" of the first order on f. In a similar manner, the neighbourhood of the second order is formed by the points of orders $k_{11}, k_{12}, k_{21}, k_{22}, \ldots$ which are indefinitely adjacent to $A'_1, A'_2, A'_3 \ldots$ on f', and so on.

The numbers $k, k_i, k_{ij}, \ldots$ obtained by the coincidence of multiple points depend on the succeeding but not on the preceding transformations.

264. Cramer used the form $y=vx$ * in particular cases, and showed that certain singularities with coin-

* Newton calls the curve obtained by the transformation $y=xy$ a *hyperbolism* of the original curve—Enumeratio linearum tertii Ordinis

cident tangents occur as final forms of singularities with distinct tangents involving a number of ultimately vanishing loops. Thus the cusp appears as a node with a vanishing loop. Nöther * also uses the same form in developing the analytical theory without any reference to geometry, which consequently fails to show clearly the existence of the various elements of the compound singularities. This defect is, however, removed by using the geometrical method of Quadric Inversion.

Ex. Consider the singularity at the origin on the curve

$$x^5 = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \quad \dots \quad (1)$$

Cramer (p. 636) uses the transformation ($y = ux$) in the form

$$x = x_1, \qquad y = x_1y_1$$

This gives a new curve $x_1 = a + by_1 + cy_1^2 + dy_1^3 + ey_1^4 \quad \dots \quad (2)$
Referring both the curves to the same axes, corresponding points can be easily constructed. The curve (2) meets the axis of y where $x_1 = 0$, and consequently from equation (1) we obtain $y^4 = 0$, which gives the four correspondents of the points where $x_1 = 0$ meets (2), *i.e.*, an arc of (2) cut off by $x_1 = 0$ corresponds to a loop of (1) closed at the origin. If, however, the four values of y_1 obtained from (2) are equal, the curve (2) has a "*serpentement*" of the appearance of ordinary contact, and (1) has the appearance of a simple cusp at the origin, which contains three vanishing loops and is really a quadruple point.

265. Practical Applications :

We shall now show by means of a few illustrative examples how, by the use of successive quadric transformations, a compound singularity may be resolved into ordinary singular points with distinct tangents. For further information and details, the student is referred to the original papers on the subject quoted before.

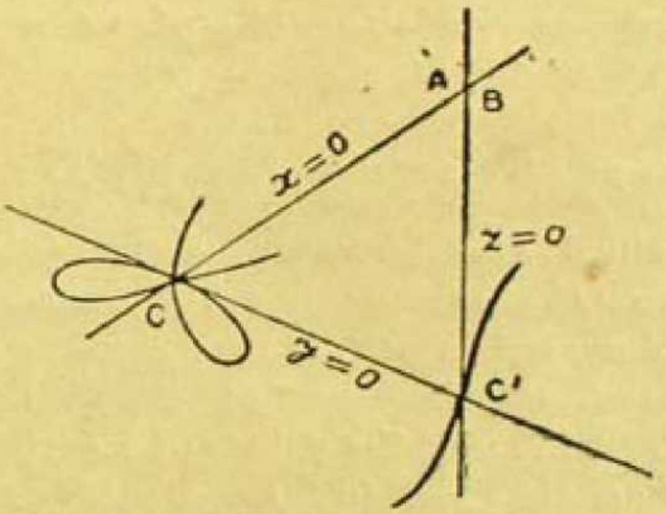
(1706). Cramer uses the same transformation—Analyse des Lignes Courbes (1750).

* Nöther—Math. Ann. Bd. 9 (1876), pp. 166-182.

Ex. 1. Discuss the nature of the singularity at the point $C(x, y)$ on the curve
$$y^3z = x^4.$$

Apply the transformation $x : y : z = x'z' : y'z' : x'^2$ (§ 218).

The transformed curve becomes $x'^3y'^3z'^3 = x'^4z'^4$, and consequently the proper inverse is $y'^3 = x'^2z'$, and the singular point C is now transformed to C' (y', z').



The nature of the point C', by writing $x' = 1$ in the equation of the transform, is obtained from the equation $y'^3 = z'$.

Thus there is an inflexion at C' on the transformed curve, with the line z'(AB) as the tangent. Therefore the singularity at C on the original curve is a triple point (§ 213).

Ex. 2. Consider the curve $y^2 = x^5$.

Applying Nöther's transformation $x = x_1$, $y = x_1y_1$, the transformed curve is
$$y_1^2 = x_1^3.$$

The transformation $x_1 = x_2$, $y_2 = x_2\, y_2$ gives for the second transform $y^2_2 = x_2$, which is a parabola and is unicursal. Consequently the original curve is also unicursal. In fact, the first transform $(x_1^3 = y_1^2)$ has a cusp at (x_1, y_1) with $y_1 = 0$ for the tangent. Hence the singularity at (x, y) is a complex singularity with two inflexions and a cusp at the origin, and there is an inflexion at infinity.

Ex. 3. The curve $y^3 = x^5$ has a triple point at the origin having an apparent appearance of a point of inflexion. This can be shown by applying two transformations successively. Thus $x = x_1$, $y = x_1\, y_1$ gives $y_1^3 = x_1^2$, which again by the transformation $y_1 = y_2\ x_1 = x_2 . y_2$ is reduced to $x_2^2 = y_2$ which is a parabola with the tangent $y_2 = 0$ (see the figure, *Ex.* 1, § 219).

Ex. 4. Take the curve $x^4 = y^7$.

This curve has a compound singularity at the point $C(x, y)$. The first transform $(x = x_1 y_1, y = y_1)$ is $x_1^4 y_1^4 = y_1^7$, *i.e.*, $x_1^4 = y_1^3$.

Here $y_1 = 0$ is the tangent at the triple point (x_1, y_1). Hence we put $x_1 = x_2$, $y_1 = y_2 x_2$, and the second transform is $y_2^3 = x_2$, *i.e.*, the point (x_2, y_2) is a point of inflexion, and consequently (x_1, y_1) is a triple point with two evanescent loops, and (x, z) is therefore a quadruple point with a neighbouring triple point.

266. Linear and Superlinear Branches:

If, as explained above, we apply successive quadric transformations so that ultimately a *k-ple* point A on f is resolved into ordinary points P, Q,...on a curve ϕ, then to the points in the neighbourhood of one of these points P, Q,...on ϕ correspond on f the points of a certain domain about A, which are then said to form a "branch"* with A as origin.

By means of a birational transformation of f, any branch is transformed into another. The principles are expressed in the following theorem:

The co-ordinates (x, y) of the points of a branch can be expressed in series of positive integral powers of a parameter t, which is again a rational function of those co-ordinates. All points of the branch will be obtained by using Gauss-plane (Argand's Diagram), if t is allowed to move within the circle of convergence of the power series.

The principal properties of the branches are obtained by considering the intersections of a branch with an algebraic curve passing through its origin A. For simplicity, we take the origin of co-ordinates at A, which therefore corresponds to the value 0 of the parameter t.

* Halphen calls it a *Cycle*.

Let F $(x, y)=0$ be an algebraic curve through A. If in this equation we now put for x and y two power series with the argument t, then the exponent (>0) of the least power of t denotes the number of intersections of the branch with the curve F.

In general, the number α of intersections of a straight line through A with any branch is called the *order* of the branch. A branch is said to be *linear* or *superlinear*, according as $\alpha=1$ or >1. If, however, any line l through A meets the branch in $\alpha+\alpha'(\alpha'>0)$ points, it is called the *singular* tangent to the branch at A.

Taking this line for the axis of x, the branch can be represented in the form :

$$x=at^{\alpha}+\ldots,\qquad y=a't^{\alpha+\alpha'}+\qquad\ldots\quad(1)$$

where a, a',...are constants different from 0. The number α' is called the *class* of the branch.

The tangent at a point of the branch is reciprocal to the point, and the numbers α and α' corresponds reciprocally. α' is the number of tangents coinciding with l, which pass through any point of l (other than A), $\alpha+\alpha'$ is the number of tangents coinciding with l which pass through A.*

Halphen† states these facts in the following theorem :

If a variable line is indefinitely near the origin of a cycle, among the points of intersection with the curve, there are points indefinitely near this origin belonging to the cycle. The number of such points is the *order* of the cycle, when the line makes a finite angle with the tangent.

* Cayley gave this theorem in Quarterly Journal, Vol. 7 (1866), p. 212, but the proof was supplied by Halphen—Comp. Rendus, t. 78 (1874), p. 1105, and by Stolz—Math. Ann. Bd. 8 (1875), p. 415. Segre gave a geometrical proof—Introduzione, etc., n'. 43.

† Halphen—E'tude sur les points singuliers, § 7.

On the other hand, if the line does not differ from he tangent, this number is equal to the sum of the order and the class.

From (1) y can be expressed in a series of positive integral powers of $x^{\frac{1}{a}}$, where in each term $x^{\frac{1}{a}}$ is to be replaced by its a values, *i.e.*, in the form: $(\omega^a=1)$

$$y=a\left(\omega x^{\frac{1}{a}}\right)^{\beta_1}+b\left(\omega x^{\frac{1}{a}}\right)^{\beta_2}+c\left(\omega x^{\frac{1}{a}}\right)^{\beta_3}+\ldots$$

The branch is said to be quadric, etc., according as $a=2, 3, \ldots$ The expression for y has precisely a values obtained by putting for $x^{\frac{1}{a}}$ each of its a values. The branches corresponding to these a values are termed by Cayley " Partial Branches." *

If the branch is real, the co-efficients in (1) are real, and real values of t correspond to real points on the branch. Hence it follows that there are the four following types of branches, according to the nature of the point A on the branch: †

(1) a odd, a' odd, A is an ordinary point,

(2) a odd, a' even, A is an inflexion,

(3) a even, a' odd, A is a cusp of the first species,

(4) a even, a' even, A is a cusp of the second species.

Therefore, from general considerations of nodes and cusps, we obtain the following:

A node is of the *k-th* species, if it has two simple points in its neighbourhood of the *k-th* order, and consequently

* One method of finding these expansions was given by Newton—Analysis per equationes numero terminorum infinitas (1669). Also Puiseux—Liouville, *t.* 15 (1), (1850), p. 365, and t. 16 (1851), p. 228.

† Stolz—Math. Ann. Bd. 8 (1875), p. 433.

it can be analysed into simple points by k quadric transformations. A cusp, on the other hand, is of the *k-th* species, if it has one simple point in its neighbourhood of the *k-th* order, and consequently it can be transformed into an ordinary point by k quadric transformations. Each node is the origin of two branches of the first order, and a cusp that of one branch of the second order.

Halphen has studied the properties of cycles and their expansions in E'tude sur les points singuliers, Part I, pp. 540-557.

267. Application of the Method of Expansions:

If, instead of the x-axis, the line $y=mx$ is a tangent, the expansion for a branch of order $\alpha \leqq k$ is obtained in the form:

$$y=mx+A_1\left(\omega x^{\frac{1}{\alpha}}\right)^{\beta_1}+A_2\left(\omega x^{\frac{1}{\alpha}}\right)^{\beta_2}+\ldots \equiv A, \text{ (say)}$$

where the origin is a *k-ple* point on the curve, and $m \neq 0$.

Then, β's are all positive integers in ascending order of magnitude and α is the least common multiple of all the denominators $(\beta_i > \alpha)$ and ω is any one of the α roots of $x^\alpha=1$. Negative exponents can, however, occur in this expansion, if the axis of y meets the curve at infinity, but we exclude that case from our discussion.

The entire portion of the curve near the origin, obtained by putting for ω every *α-th* root of unity in turn, is then called a *superlinear branch* of order α, having $y=mx$ for the tangent. The different superlinear branches of a curve at a *k-ple* point give in all $\Sigma\alpha=k$ expansions, but the individual branches can have different or any number of common tangents at the point, *i.e.*, in the expansions

of the branches, the co-efficients A_1, A_2,...may, some or all of them, be identical, or any two or more expansions can be identical up to a certain finite number of terms. In this case we must take the expansion until the two branches separate.

It follows then that if a point is an ordinary point on the curve, only one *linear* branch (the curve itself) passes through it. If it be an ordinary *k-ple* point with distinct tangents, there pass k linear branches through the point. If, however, two or more tangents coincide, we have *superlinear* branches.

The following illustration will clearly explain how expansions are useful in studying the nature of higher singularities :

Ex. Consider a Ramphoid cusp, *i.e.*, a cusp of the second species at the origin on a curve.

There is a superlinear branch of order $a=2$, whose expansion may be written as :

$$y=mx+A_1\omega^4x^{\frac{4}{2}}+A_2\omega^5x^{\frac{5}{2}}+A_3\omega^6x^{\frac{6}{2}}+\ldots \qquad \ldots \quad (1)$$

where $y=mx$ is the tangent, $a=2$ and $\omega=\pm 1$.

Hence the expansion (1) becomes

$$y=mx+A_1x^2\pm A_2x^{\frac{5}{2}}+A_3x^3\pm A_4x^{\frac{7}{2}}+\ldots$$

If the origin is a cusp of the first species, the expansion of the branch (of order $a=2$) is—

$$y=mx\pm A_1x^{\frac{3}{2}}+A_2x^2\pm A_3x^{\frac{5}{2}}+\ldots \qquad \ldots \quad (2)$$

which evidently differs from (1) only by the term $x^{\frac{3}{2}}$.

268. Practical Method :

The above expansions and other connected formulæ are proved in works on the Theory of Functions to which branch they properly belong. Without going further into the details of the theory, we shall now exhibit how these expansions are obtained in actual practice for determining the intersections of curves at higher singular points.

Ex. 1. Find an expansion for the branch near the origin of the curve

$$y-x^2-x^2y+2y^3=0.$$

Assume $$y=ax+bx^2+cx^3+dx^4+\dots$$

Substituting this expansion for y in the given equation, we obtain

$$ax+bx^2+cx^3+\dots-x^2-x^2(ax+bx^2+cx^3+\dots)$$
$$+2(ax+bx^2+cx^3+\dots)^3=0,$$

i.e., $$ax+(b-1)x^2+(c-a+2a^3)x^3+(d-b+2a^2b)x^4+\dots=0.$$

Equating the co-efficients of x, x^2, x^3, x^4, ...to zero, we find

$$a=0,\quad b=1,\quad c=a-2a^3=0,\quad d=b-2a^2b=1, \text{ and so on.}$$

$\therefore$ The required expansion for the branch is—

$$y=x^2+x^4+\dots$$

Ex. 2. Expand $(y+x^2)^2=4y^2(x^2+y^2)$ near the origin.

Newton's diagram gives for the first approximation $y+x^2=0$.

The next approximation is given by $y+x^2=2y\sqrt{x^2+y^2}$.

$\therefore$ Assuming $y=-x^2+ax^3+bx^4+\dots$, we obtain—

$$(ax^3+bx^4+\dots)^2=4(-x^2+ax^3+bx^4+\dots)^2\{x^2+(-x^2+ax^3+\dots)^2\}$$

whence $$a^2=4,\ 2ab=-8a,\ b^2+2ac=4+4a^2.$$

i.e., $$a=\pm 2,\quad b=-4,\quad c=\pm 9,\quad \text{etc.}$$

$$\therefore\quad y=-x^2\pm 2x^3-4x^4\pm 9x^5\pm\dots$$

Ex. 3. Find expansions for the branches of the following curves near the origin :

(*i*) $y=xy+2x^2+3x^2y$ (*ii*) $y=x^2+xy^3$

(*iii*) $(y-x^3)^2=x^2y^2$.

269. Expansion of a Function:

Let the implicit function $F(x, y)$ be of order n, and suppose that the origin is a *k-ple* point and that neither axis is a tangent to the curve.

Let $$y=\phi_1(x),\quad y=\phi_2(x),\quad \dots\quad y=\phi_k(x)$$

be the expansions corresponding to the k linear branches

at the *k-ple* point. These are, in fact, the expansions for k of the n values of y obtained by putting $x=0$ in the equation $F(x, y)=0$. Besides the singular point, the curve intersects the y-axis in $n-k$ other finite and distinct points, for each of which there is a similar expansion, having a constant added to it. This constant evidently represents the distance of the point from the origin.

Thus, we obtain $(n-k)$ expansions of the form—

$$y = B_0 + B_1 x + B_2 x^2 + \dots \equiv \psi,$$

which is an ordinary power series, representing the implicit function $F(x, y)$.

These series are all convergent within a certain region, and indeed are absolutely convergent when each term is replaced by its absolute value.* The product of two such series is also absolutely convergent,† and consequently, we may represent the equation of the curve as the product of such series:

$$\Pi(y-\phi).\Pi(y-\psi) \equiv F(x, y).$$

270. Discriminantal Index:

We have seen in § 82 that the first polar of any point passes through the points of contact of tangents drawn from that point to the curve and the class of a curve is determined by the number of intersections of the curve with its first polar, which, however, is reduced by the existence of nodes and cusps (§ 121). Thus, for determining the effect of higher singularities on the class of a curve, we require to find the number of intersections of the curve with its first polar at such a singular point.

* Townsend—Functions of a Complex Variable, § 47.

† Cauchy—Analyse Algebrique, Chap. VI.

Let $F=0$ be the equation of an n-*ic* of class m, and consider its intersections with the first polar of any point, (0, 1, 0) for example. If, therefore, we eliminate y (say) between $F=0$ and the equation $\partial F/\partial y=0$ of the first polar, we obtain the resultant in the form of an equation $\Theta(x)=0$, where $\Theta(x)$ is, in general, of order $n(n-1)$ in x.

Since the class of the curve is m, the equation $\Theta(x)=0$ will have m simple roots x' corresponding to the points of contact of the m tangents which can be drawn from the point to the curve. Dividing the equation by these m factors $\Pi(x-x')$, the remaining factor $\Theta(x)/\Pi(x-x')$ is of degree $n(n-1)-m$, and when equated to zero gives $n(n-1)-m$ roots, which correspond to the intersections at the singular points. Thus the singular points count as $n(n-1)-m$ intersections of the curve with its first polar, and when there are only nodes and cusps on the curve and their numbers are δ and κ respectively, they count as $2\delta+3\kappa$ intersections (§ 121).

The number $J\equiv n(n-1)-m=2\delta+3\kappa$ is called the total "Discriminantal Index" of the singularity.

The roots of the equation $\Theta(x)=0$ give the so-called *branch points* of the function. The root with the least modulus gives the limit of convergence of the above series, which, therefore, converges absolutely up to the nearest branch-point. But the discriminant of F is the product $\Pi(y_i-y_r)^2$ of the squared differences of the roots of $F=0$ regarded as an equation in y. Therefore, within this convergence limit, we can consider the discriminant $\Theta(x)$ equal to the product of the squared differences of the series ϕ and ψ taken *two* at a time.

271. From what has been stated above, the reduction in the class of a curve due to the existence of higher singularities can now be easily determined.

The discriminant $\Theta(x) \equiv \Pi(\phi - \psi)^2$ can be divided into two parts—one part, called the "variable" factor corresponding to the m proper tangents, and the other the "fixed" factor of order $n(n-1)-m$ corresponding to the singular point. This fixed factor therefore is the product $\Pi\,(\phi_i - \phi_r)^2$ of the $\frac{1}{2}k(k-1)$ differences corresponding to the k partial branches through the point taken two at a time. The number of roots of the equation $\Pi(\phi_i - \phi_r)^2 = 0$ is therefore equal to *twice* the number of intersections of all the partial branches with one another, each root being counted twice, *i.e.*, equal to twice the number of intersections of the curve with itself at the singular point.

But twice the number of intersections is equal to the discriminantal index $J = 2\delta + 3\kappa$. For the discriminant $\Theta(x)$ is the result of eliminating y between $F=0$ and $\partial F/\partial y = 0$. Therefore the number J is equal to twice the number of intersections of the curve with its first polar at the singular point.

Combining the results obtained, we may state the following theorem:*

The number $J = 2\delta + 3\kappa$, which represents the reduction in the class of a curve due to the existence of higher singularities is equal to twice the number of intersections of the curve with itself at the singular point.

Ex. Consider the curve $\quad x^5 + y^5 - 4x^3y + y^3 = 0.$

The origin is a triple point consisting of a cusp and two nodes and there is a third node. $\quad \therefore \ \delta = 3, \quad \kappa = 1.$

There are two expansions:—

$$y = \pm 2x^{\frac{3}{2}} + \dots \qquad \text{and} \qquad y = \tfrac{1}{4}x^2 + \dots$$

Hence $J =$ twice the number of intersections $= 2.\frac{3}{2}.3 = 9 = 2\delta + 3\kappa.$

* Halphen—Bulletin de la Soc. Math. de France, Vol. I, p. 133. Also Zeuthen—"Sur les singularities des courbes planes," Math. Ann Bd. 10 (1876), p. 213.

272. Intersections of Two Curves at a Singular Point :

From what has been said above with regard to the intersections of a curve with its first polar at a singular point, we can easily determine them for the general case of any two curves, *i.e.*, how many of their intersections coincide at the singular point.

Let $F_n(x, y)=0$ and $F_n'(x, y)=0$ be the equations of any two curves of orders n and n', having at the origin multiple points of orders k and k' respectively.

Assuming as before, the curve $F_n=0$ admits of $n-k$ expansions of the type ψ and k expansions of the type ϕ, so that

$$F_n \equiv \Pi(y-\phi)\times\Pi(y-\psi)=0 \qquad \ldots \quad (1)$$

Similarly, $F_n'=0$ admits of $n'-k'$ and k' expansions of the types ϕ' and ψ' respectively, so that

$$F_n' \equiv \Pi(y-\phi')\times\Pi(y-\psi')=0 \qquad \ldots \quad (2)$$

If we eliminate y between (1) and (2) we obtain the resultant $\Theta(x)$, of order nn' in x, which may be written in the form of the difference-product

$$\Theta(x)\equiv\Pi(\phi-\phi')\times\Pi(\phi-\psi')\times\Pi(\psi-\phi')\times\Pi(\psi-\psi')=0$$

The zero roots of this equation will give the intersections at the origin.

The factors $\Pi(\phi-\phi')$ of the kk' differences all contain x in a certain power λ as a factor; and if we proceed with all the differences, we obtain

$$L_{x=0}\frac{\Pi(\phi-\phi')}{x^{\Sigma\lambda}}=\text{constant}.$$

Accordingly we obtain the number of intersections of two curves at a singular point by adding together the

different values of λ corresponding to the kk' partial branches each defined by the equation

$$L_{x=0} \frac{y-y'}{x^{\lambda}} = \text{constant},$$

i.e., it is equal to the sum of the numbers of intersections of one curve with the branches of the other.*

We may state the above facts in the following theorem :—

If $y=\phi_i$ and $y=\psi_i$ be the expansions of y in terms of x for branches of two curves passing through the singular point at the origin, the number of intersections of the two curves which coincide at the origin is equal to the index of the product of the lowest powers of x in all possible expressions of the form $\phi_i - \psi_i$.

Ex. 1. Consider a curve with a cusp at the origin, and another passing through the cusp and touching the first curve there.

Here the expansions for the two curves are respectively :

$$\left.\begin{aligned} y &= ax^{\frac{3}{2}} + bx^2 + \dots &= \phi_1 \\ y &= -ax^{\frac{3}{2}} + bx^2 - \dots &= \phi_2 \end{aligned}\right\} \qquad \dots \quad (1)$$

$$y = \alpha x^2 + \beta x^3 + \dots \qquad = \psi_1 \qquad \dots \quad (2)$$

Then, $\phi_1 - \psi_1 = x^{\frac{3}{2}}(a - \alpha x^{\frac{1}{2}}) + \dots$ and $\phi_2 - \psi_1 = x^{\frac{3}{2}}(\alpha x^{\frac{1}{2}} - a) + \dots$

Hence ψ_1 meets each of ϕ_1 and ϕ_2 in $\frac{3}{2}$ points, and the total intersections $= \frac{3}{2} \times 2 = 3$.

Ex. 2. Consider the intersections of a curve with its Hessian at a double point.

Let $F = (y - mx)(y - px) + u_3 + \dots = 0$ be the equation of a given curve, having a node at the origin.

* Halphen—Bull. de la Soc. Math. de France, t. 4 (1875), p. 59. Journal de Math.—Bd. 2(3) (1876), pp. 257, 371.

See also Brill—Sitzungsberichte der math. phys. Klasse, etc., zu München, Vol. 18 (1888), p. 81.

It has at that point the two expansions :—

$$y_1 = mx + m_1x^2 + \dots \;= \phi_1$$

$$y_2 = px + p_1x^2 + \dots \;= \phi_2$$

The Hessian has at this point a double point with the same tangents. Therefore, for the Hessain we have—

$$y_1' = mx + a_1x^2 + \dots \;= \psi_1$$

$$y_2' = px + b_1x^2 + \dots \;= \psi_2$$

whence it is easily seen that the differences $\phi_1 - \psi_2$ and $\phi_2 - \psi_1$ each gives one intersection, while the differences $\phi_1 - \psi_1$ and $\phi_2 - \psi_2$ each gives two intersections, since the branches touch.

Thus, we get altogether $1 + 1 + 2 + 2 = 6$ intersections, as was otherwise found in § 103.

At a cusp, however, the curve has the expansions—

$$y_1 = mx^{\frac{3}{2}} + \dots = \phi_1 \qquad y_2 = -mx^{\frac{3}{2}} + \dots = \phi_2.$$

The expansions for the Hessian are—

$$y_1' = mx^{\frac{3}{2}} + \dots \equiv \psi_1 \qquad y_2' = -mx^{\frac{3}{2}} + \dots \equiv \psi_2$$

$$y_3' = ax + bx^2 + \dots \equiv \psi_3 \text{ (for the ordinary branch).}$$

The third branch ψ_3 intersects the two partial branches ϕ_1 and ϕ_2 in two points (each partial branch once), while the four partial branches ϕ_1, ϕ_2, ψ_1, ψ_2 meet each other in $\frac{3}{2}$ points. Thus we obtain the total number of intersections $= \frac{3}{2} \times 4 + 2 = 8$.

Ex. 3. Consider a curve with a simple cusp at the origin and another with a cusp of the second species, having the same tangent. The expansion for the first curve is $\qquad y = x^{\frac{4}{3}} + \dots$
which is a superlinear branch of order 3, consisting of the partial branches—

$$\left.\begin{aligned} y &= x^{\frac{4}{3}} + \dots &&= \phi_1 \\ y &= \omega x^{\frac{4}{3}} + \dots &&= \phi_2 \\ y &= \omega^2 x^{\frac{4}{3}} + \dots &&= \phi_3 \end{aligned}\right\} \text{ where } \omega^3 = 1 \;\dots (1)$$

For the ramphoid cusp, we have $y=x^2 \pm x^{\frac{5}{2}}+\dots$

i.e.,
$$\left.\begin{aligned} y&=x^2+x^{\frac{5}{2}}+\dots=\psi_1 \\ y&=x^2-x^{\frac{5}{2}}+\dots=\psi_2 \end{aligned}\right\} \qquad \dots \quad (2)$$

Each of the branches ψ_1 and ψ_2 meets the partial branches ϕ_1, ϕ_2, ϕ_3 in $\frac{4}{3}$ points,

since
$$\phi_i-\psi_j=x^{\frac{4}{3}}(\alpha-1)+\dots \qquad \alpha=1,\ \omega,\ \omega^2.$$

Hence the total intersections $=\frac{4}{3}\times 3+\frac{4}{3}\times 3=8$.

Ex. 4. Consider two curves having respectively k and k' linear branches through the origin, with all distinct tangents.

In this case the product $\Pi(\phi_i-\psi_j)$ contains kk' factors of the type
$$ax+bx^2+cx^3+\dots$$

Hence the curves meet in kk' points at the origin. If, however, the curves have the same tangents, the product $\Pi(\phi_i-\psi_j)$ contains $k(k-1)$ factors of the type $ax+bx^2+cx^3\dots$ and k factors of the type
$$b'x^2+c'x^3\dots$$

Hence the curves meet $k(k-1)+2k=k(k+1)$ times at the origin.

It follows, in particular, that the tangent to the superlinear branch of § 267 meets it in β_1 points.

273. Expansions in Line Co-ordinates:

Just as a curve regarded as a locus of points has point singularities, regarded as an envelope of lines, it has line singularities. In order to examine this closely, we require to obtain an expansion in line co-ordinates of the branch of a curve near the singular point.

We shall suppose that the axis of x, and not the y-axis, is a tangent at the origin, which is supposed to be a singular point. Therefore no proper fraction will appear in the expansion of the branch in point co-ordinates, which may be written in the form:

$$y=a_1x^{\frac{\beta_1}{\alpha}}+a_2x^{\frac{\beta_2}{\alpha}}+\dots \quad (\beta_i>\alpha) \qquad \dots \quad (1)$$

where β_1 denotes the number of points which the tangent has in common with the branch at the origin. We may write the equation of the tangent at any point of the curve in the form $\xi x+y+\zeta=0$, which expresses the fact that the point $(x, y, 1)$ lies on the line $(\xi, 1, \zeta)$. The co-ordinates of the tangent are given by—

$$\xi=-\frac{\partial y}{\partial x},\ \ \zeta=x\frac{\partial y}{\partial x}-y \qquad \dots \qquad (2)$$

If the axis of x, *i.e.*, the line $y=0$ is the tangent, we have $\xi=0$, $\eta=1$, $\zeta=0$.

We shall now find an expansion of ζ in terms of ξ in the form—

$$\zeta=A_1\xi^{\frac{\beta_1'}{\alpha'}}+A_2\xi^{\frac{\beta_2'}{\alpha'}}+ \qquad \dots \qquad (3)$$

The least common multiple α' of the denominators in the exponents of this series is called the *order* of the branch regarded as an envelope of tangents, *i.e.*, its *class*, and denotes the number of tangents drawn to the curve from a point on the singular tangent and coinciding with it (excluding the tangent to the other branch, if any). β_1' is the number of tangents passing through the singular point and coinciding with the singular tangent.

From (1) we obtain

$$\xi=-\frac{\partial y}{\partial x}=-a_1.\frac{\beta_1}{\alpha}.x^{\frac{\beta_1-\alpha}{\alpha}}- \qquad \dots \qquad (4)$$

and $$\zeta=(a_1\frac{\beta_1}{\alpha}.x^{\frac{\beta_1}{\alpha}}+\dots)-(a_1x^{\frac{\beta_1}{\alpha}}+\ \dots\ )$$

$$=-a_1.\frac{\beta_1-\alpha}{\alpha}.x^{\frac{\beta_1}{\alpha}}+\dots \qquad \dots \qquad (5)$$

Eliminating x between (4) and (5), we may obtain the required relation between ζ and ξ. Now, from the series (4) we can deduce a series expressing $x^{\frac{1}{\alpha}}$ in powers of $\xi^{\frac{1}{\beta_1-\alpha}}$ and put this in (5). Thus we obtain—

$$\zeta = A_1 \, \xi^{\frac{\beta_1}{\beta_1-\alpha}} + \ \dots \qquad \dots \qquad (6)$$

This series must be the same as the series (3).

Since we have taken only the first term from the series (4), we cannot at once say that $\alpha' = \beta_1 - \alpha$, or, that to a given value of ξ, $\beta_1 - \alpha$ values of ζ correspond. For, it may happen that in (6) all terms having the least common denominator are absent. But in every case it is certain that

$$\frac{\beta_1'}{\alpha'} = \frac{\beta_1}{\beta_1 - \alpha}, \quad \alpha' \leqq \beta_1 - \alpha, \text{ and therefore, } \beta_1' \leqq \beta_1.$$

On the other hand, however, we may consider the series (1) deduced from (6) exactly as the series (6) was deduced from (1), and in that case we must have $\beta_1 \leqq \beta_1'$.

It follows then that $\beta_1' = \beta_1$, $\alpha = \beta_1' - \alpha'$, $\alpha' = \beta_1 - \alpha$. Combining all these results we may state the following theorem :—*

The number β_1 of the coincident intersections of the tangent of a superlinear branch is equal to the number β_1' of the tangents at the singular point which coincide with the singular tangent. If α and α' are the order and class respectively of the branch, then always

$$\alpha + \alpha' = \beta_1 = \beta_1'.$$

* Zeuthen—" Note sur les singularities des courbes planes "—Math. Ann. Bd. 10 (1876), pp. 210-220.

If there are more than one branch passing through the singular point, having the same tangent, the result can be obtained by addition of the results just established for a single branch.

Ex. 1. Consider the cusp on the curve $y=x^{\frac{4}{3}}$.

We have $\xi=-\dfrac{\partial y}{\partial x}=-\dfrac{4}{3}x^{\frac{1}{3}}$ and $\zeta=\dfrac{4}{3}x^{\frac{4}{3}}-x^{\frac{4}{3}}=\dfrac{1}{3}x^{\frac{4}{3}}$

$\therefore$ Eliminating x, we obtain $\zeta=\dfrac{3^3}{4^4}\xi^4$

$$\therefore \qquad \alpha'=\beta_1-\alpha=4-3=1$$

i.e., there is only a single tangent, and the cusp of the first species is only a point-singularity.

Ex. 2. Consider the cusp of the second species.

The expansion for the branch is of the form :

$$y=x^2 \pm x^{\frac{5}{2}}+x^3 \pm \ldots \qquad (\S\ 266)$$

Now, taking the branch $\qquad y=x^2+x^{\frac{5}{2}}+x^3+\ldots \qquad \ldots \quad (1)$

we obtain
$$\xi=-\frac{\partial y}{\partial x}=-2x-\tfrac{5}{2}x^{\frac{3}{2}}+\ldots \qquad \ldots \quad (2)$$

$$\zeta=(2x^2+\tfrac{5}{2}x^{\frac{5}{2}}+\ldots)-(x^2+x^{\frac{5}{2}}+x^3+\ldots)=x^2+\tfrac{3}{2}x^{\frac{5}{2}}+\ldots \quad (3)$$

Now, we shall have to eliminate x between (2) and (3), and in doing this we have to express x in terms of ξ by the process known as *reversion of series.*

Let
$$x^{\frac{1}{2}}=a_1\xi^{\frac{1}{2}}+b_1\xi+c_1\xi^{\frac{3}{2}}+\ldots$$

where $a_1, b_1, c_1, \ldots$ are to be determined from the series (2).

Substituting in (2) we obtain—

$$\xi=-2(a_1\xi^{\frac{1}{2}}+b_1\xi+\ldots)^2-\frac{5}{2}(a_1\xi^{\frac{1}{2}}+b_1\xi+\ldots)^3+\ldots$$

$$=-2a_1^2\xi-(4a_1b_1+\tfrac{5}{2}a_1^3)\xi^{\frac{3}{2}}+\ldots$$

Equating the co-efficients on both sides, we get—

$$a_1^2=-\tfrac{1}{2}, \qquad \text{and} \qquad 4a_1b_1+\tfrac{5}{2}a_1^3=0$$

whence $\qquad a_1=\dfrac{i}{\sqrt{2}}$ and $b_1=\dfrac{5}{16}$.

$$\therefore \qquad x^{\frac{1}{2}}=\frac{i}{\sqrt{2}}\,\xi^{\frac{1}{2}}+\frac{5}{16}\,\xi+\dots \qquad \dots \quad (4)$$

Substituting this value in (3), we obtain—

$$\zeta=\left(\frac{i}{\sqrt{2}}\,\xi^{\frac{1}{2}}+\frac{5}{16}\,\xi+\dots\right)^4+\frac{3}{2}\left(\frac{i}{\sqrt{2}}\xi^{\frac{1}{2}}+\frac{5}{16}\,\xi+\dots\right)^5+\dots$$

$$=\frac{\xi^2}{4}+\left(4\,\frac{i^3}{2\sqrt{2}}\cdot\frac{5}{16}+\frac{3}{2}\cdot\frac{i^5}{4\sqrt{2}}\right)\xi^{\frac{5}{2}}+\dots$$

$$=\frac{\xi^2}{4}-i\,\frac{\xi^{\frac{5}{2}}}{4\sqrt{2}}+\dots$$

$$=\left(\frac{\xi}{2}\right)^2-i\left(\frac{\xi}{2}\right)^{\frac{5}{2}}+\dots$$

where, it is seen, only negative values of ξ give real values of ζ. Hence, putting $-\xi$ for $\dfrac{\xi}{2}$, we get the reciprocal expansion in the form

$$\zeta=\xi^2+\xi^{\frac{5}{2}}+\dots$$

274. Polar Reciprocal of a Superlinear Branch :*

If, in the expansion (3) of § 273, we put x and y for ξ and ζ respectively, we obtain the equation of the polar reciprocal of the branch with respect to the conic $x^2+2y=0$ (§ 114). Thus the polar reciprocal of the branch is—

$$y=A_1x^{\frac{\beta_1'}{\alpha'}}+A_2x^{\frac{\beta_2'}{\alpha'}}+\dots$$

which is again a superlinear branch of order α', and is met by the tangent in β_1' points coinciding with the origin.

* *Cf.* Hilton—Plane Alg. Curves, Chap. VI, § 5.
See also A. B. Basset—Quart. J. of Math., Vol. 45 (1914), pp. 52-65.

Hence, we may state the theorem:

If the polar reciprocal of a superlinear branch of order α *at any point O whose tangent meets it in* β_1 *points coinciding with O is a superlinear branch of order* α' *at any other point O′ whose tangent meets it in* β_1' *points coinciding with O′, then*

$$\alpha+\alpha'=\beta_1'=\beta_1.$$

Again, by the properties of reciprocal singularities, the number of tangents drawn to a superlinear branch at O, coinciding with the singular tangent at O, is equal to the number of intersections of the reciprocal branch with its tangent coinciding with the singularity O′. The number of tangents from a point on the tangent at O coinciding with this tangent is equal to the number of intersections of a line through O′ with the reciprocal branch coinciding with O′.

Hence, from the above relations, *i.e.*, $\alpha+\alpha'=\beta_1=\beta_1'$, we at once deduce the following theorem:

If a superlinear branch of order α *at O meets its tangent in* β_1 *points coinciding with O, then* β_1 *tangents from O to the branch coincide with the tangent at O and* $\beta_1-\alpha$ *tangents to the branch from any point of the tangent at O coincide with the tangent at O.*

Since the origin is a *k-ple* point, the tangent $y=0$ meets the curve in $k+1$ points at the origin. But again there is a superlinear branch of order α, and consequently the line $y=0$ meets the other branches in $k-\alpha$ points at the origin. Hence it meets the superlinear branch in $\alpha+1$ points.

$$\therefore\quad \beta_1=\alpha+1 \qquad \text{and} \qquad \alpha'=1,\quad \beta'=\alpha+1$$

or, in other words:

The polar reciprocal of a superlinear branch of order α *is, in general, a linear branch having* $(\alpha+1)$*-pointic contact with its tangent.*

Ex. Consider the cusp of the first species at the origin with $y=0$ as the tangent.

The expansions are $\quad y=\pm ax^{\frac{3}{2}}+bx^{2}\pm cx^{\frac{5}{2}}+\ldots$

The expression in line co-ordinates, obtained by the method of § 273 becomes—

$$\zeta=-\frac{4}{27a^2}\xi^3-\frac{16b}{81a^4}\xi^4+\ldots$$

whence the polar reciprocal *w.r.t.* $x^2+2y=0$ of the branch is—

$$y=-\frac{4}{27a^2}x^3-\frac{16b}{81a^4}x^4+$$

which shows that it is a linear branch with an inflexion at the origin. Thus we have a proof of the fact that to a cusp corresponds an inflexional tangent on the reciprocal curve.

275. Cuspidal Index :

Existence of higher singular points as well reduces the deficiency of a curve. To find its effect, we have to determine an equivalent number. $\delta+\kappa$, which has the same effect on the deficiency as the singularities. We shall now find the value of κ :

Consider a curve S with a superlinear branch of order a at any point O. Let S′ be another curve of the same order and class with δ nodes and κ cusps, and having a one-to-one relation with S. If now A and A′ are two fixed points in the same plane, and P, P′ two corresponding points on S and S′ respectively, the order and class of the locus C of the intersection of AP and A′P′ can be determined by the method of § 152. The order of C is $2n$, with an *n-ple* point at each of A and A′. To determine the class, we count the number of tangents which can be drawn from A or A′. There are $2n$ tangents at A or A′, and m tangents which are tangents to S or S′. Now, consider the line AO. Since the point O on S corresponds

to a consecutive points on S′. AO has a-pointic contact with C at any point H. We have now to consider two cases: (1) the a consecutive points lie on a simple branch of S′, so that all cusps of S′ correspond to only ordinary points on S. In this case C has a singular point at H, where AH meets it in a points, but A′H in only one point. Hence, by the formula of § 274, the number of tangents coinciding with AH is $a-1$, and consequently the class of C is $2n+m+a-1$. On the other hand, to each line through A′ and a cusp of S′ corresponds a tangent to C through A′, so that the class of C is

$$2n+m+\kappa.$$

Hence, $\kappa = a-1$

(2) The a consecutive points on S′ may coincide with a cusp on S′. Then the singularity H on C is met by AH in a points, but by A′H in two points only, and consequently, $a-2$ tangents coincide with AH. Hence the class of C is $2n+m+a-2$. Again, since a line through a cusp is not a tangent, counting the number of tangents from A′, the class of C becomes $2n+m+\kappa-1$,

whence $\kappa = a-1$.

The number κ is called the "cuspidal index"* of the singularity at O, and we have—

The cuspidal index κ of a singularity is one less than the order a.†

Similarly, in the line system $\iota = a'-1$.

* Smith—Proc. Lond. Math. Soc., Vol. VI (1873-76); Nöther calls it "Verzweigung"—Math. Ann. Bd. 9 (1876), p. 166.

† For a geometrical demonstration see Bertini—Lomb. Ist. Rend., Vol. 21 (2) (1888), pp. 326-413, and Vol. 23 (1890), p. 307.

276. Extension of Plücker's Formulae :

The first polar of any point is an adjoint curve (§ 227) which has at each *k-ple* point on the curve a $(k-1)$-*ple* point. But each *k-ple* point A counts as $\Sigma k(k-1)$ intersections of a curve with any adjoint, where the sum extends over all of Nöther's component points. Therefore, for a first polar (adjoint) this number is to be increased by $\Sigma(a-1)$, where Σ extends over the orders a of the superlinear branches passing through A, and $\Sigma(a-1)$ is called the "Cuspidal Index," as explained above, of the singular point.

Hence, the class of an *n-ic* is given by (§ 146)

$$m=n(n-1)-\Sigma k(k-1)-\Sigma(a-1) \qquad \ldots \quad (1)$$

and reciprocally,

$$n=m(m-1)-\Sigma k'(k'-1)-\Sigma(a'-1) \qquad \ldots \quad (2)$$

where k' and a' represent respectively the order of the multiple tangent and the class of the superlinear branch.

Since a curve and its reciprocal have the same deficiency, we may write

$$n(n-3)-\Sigma k(k-1)=m(m-3)-\Sigma k'(k'-1) \qquad \ldots \quad (3)$$

$$\therefore \quad 2(p-1)=n(n-3)-\Sigma k(k-1)=\Sigma(a-1)+m-2n$$

$$=m(m-3)-\Sigma k'(k'-1)=\Sigma(a'-1)+n-2m,$$

whence
$$\Sigma(a-a')=3(n-m)$$

$$\left.\begin{aligned}\Sigma(2a+a'-3)&=3(n+2p-2)\\ \Sigma(a+2a'-3)&=3(m+2p-2)\end{aligned}\right\}$$

where, in the last equation, the sum extends to all the branches for which $aa'>1$.

All these investigations properly belong to the theory of functions, and without going further into the details, we shall conclude the topic by mentioning the important fact that Plücker's Formulae are applicable to curves with higher singularities, if we regard the singular point or singular tangent as equivalent to a number of nodes and cusps or bitangents and inflexional tangents respectively. The four equivalent numbers δ_1, ϵ_1, δ_1', ϵ_1',—which Zeuthen calls "Principal Equivalence," and Smith terms δ_1 and ϵ_1 "Nodal and cuspidal index"—are connected by three independent equations, to which the expression for deficiency is added, so that the four can be determined.

Thus, for a superlinear branch (a, a'), which diminishes the class by J, the discriminantal index, and the order by J′, these numbers are determined by the equations

$$\epsilon_1 = a - 1,\ \epsilon_1' = a' - 1, \qquad J = 2\delta_1 + 3\epsilon_1, \qquad J' = 2\delta_1' + 3\epsilon_1'$$

The four numbers are connected by the relation:

$$\delta_1 - \delta_1' = \tfrac{1}{2}(\epsilon_1 - \epsilon_1')(\epsilon_1 + \epsilon_1' - 1).$$

Brill * has shown that each higher singularity can be shown as limiting cases of ordinary singularities by means of a series of deformation processes, which can, however, be effected by means of quadratic transformations as Scott † has shown. Prof. Basset in a number of papers,‡ has determined the point and line constituents of certain singularities, and specially considered the resolution of multiple points with tacnodal branches, etc.

* Brill—Math. Ann. Bd. 16 (1880), p. 348.

† Scott—Am. J. of Math., Vol. 14 (1892), p. 301, and Vol. 15 (1893), p. 221.

‡ A. B. Basset—Quart. J. Math., Vol. 37 (1906), p. 313, and Vol. 43 (1912), p. 151.

For a real curve of order n and class n', with ω' real inflexions, t'' real bitangents with imaginary points of contact, r' real cusps and d'' real double points with imaginary tangents (acnodes), F. Klein * established the relation †—

$$n+\omega'+2t''=n'+r'+2d''$$

For a curve with higher singular points, see Juel—Math. Ann. Bd. 61 (1905), p. 77.

277. Curves of Closest Contact: Osculating Curves:

The process of expansions described in this Chapter affords a very convenient method of studying intersections of curves at multiple points, whether of point-singularities or line-singularities, and in particular, of studying contact of curves at any given point. At the outset, however, we may state the following theorem:

When two curves have linear branches through the origin, and the expansions of y in terms of x for the branches are identical as far as the term with x^r, the curves have $(r+1)$-pointic contact at the origin.

For in this case the difference $y_1-y_2\equiv\phi_1-\psi_1$ (§ 272) contains x^{r+1} as a factor, which shows that the curves intersect $(r+1)$ times at the origin, *i.e.*, they have $(r+1)$-pointic contact at the origin.

The same process applies to the general case of any point on a curve, the point being taken as the origin of co-ordinates.

We may use this theorem for finding curves having closest contact with a given curve possible for a curve of that order.

* F. Klein—Math. Ann. Bd. 10 (1876), p. 199.

† T. R. Hollcroft—On the Reality of Singularities of Plane Curves, Math. Ann., Vol. 97 (1927), pp. 775-787.

Let the given curve be an *n-ic*, and we require to find an *m-ic* having at the origin the closest possible contact with the given curve.

Now, since the *m-ic* is determined by $\frac{1}{2}m(m+3)$ points, only $\frac{1}{2}m(m+3)$ points can be assumed on the *n-ic* and the *m-ic* is completely determined. Hence, the *m-ic* of closest possible contact can have at the most $\frac{1}{2}m(m+3)$-pointic contact with the given curve. If, however, the *m-ic* is subjected to satisfy other conditions, the order of contact will be reduced.

Thus, a circle can have only three-pointic contact, a parabola can have four-pointic contact, and a general conic can have five-pointic contact, and so on.

Definition : Curves having closest possible contact with a given curve at a given point on the latter is called the *osculating curve* at that point. Thus, the circle of curvature at any point is the osculating circle, etc.

Ex. 1. Find the osculating conic at the origin of the curve

$$y = x + x^2 - 2x^3$$

Let $y = mx + Ax^2 + 2Hxy + By^2$ be the required conic, and

assume $$y = ax + bx^2 + cx^3 + \dots$$

$$\therefore \quad ax + bx^2 + cx^3 + \dots = mx + Ax^2 + 2Hx(ax + bx^2 + \dots) + B(ax + bx^2 + \dots)^2$$

Equating the co-efficients of x, x^2, x^3, etc., and comparing the values of a, b, c, ... with the equation of the curve, we obtain—

$$a = m = 1, \qquad b = A + 2aH + Ba^2 = A + 2H + B = 1$$

$$c = 2b(H + aB) = (2H + B) = -2 \quad \text{and} \quad H + B = -1$$

$$d = 2cH + Bb^2 + 2acB = -4H + B - 4B = 0, \ \textit{i.e.}, \ 3B + 4H = 0$$

$$\therefore \quad A = -1, \ B = -4 \ \text{and} \ H = 3$$

whence the equation of the osculating conic is—

$$y = x - x^2 + 6xy - 4y^2$$

If, however, the conic is to be a parabola, $AB=H^2$, whence

$$A=4, \quad H=-2 \quad \text{and} \quad B=1.$$

and the equation of the osculating parabola is—

$$y=x+4x^2-4xy+y^2=x+(2x-y)^2.$$

Ex. 2. Find the conics of closest contact at the origin of the curve

$$x^3+y^3=3axy.$$

Here, the two branches of the curve have the axes of co-ordinates as tangents, and there will be two osculating conics for the two branches.

Consider the branch with $y=0$ as tangent, and let its expansion be

$$y=a'x^2+b'x^3+\ldots$$

Substituting in the equation of the curve, we have—

$$x^3+(a'x^2+b'x^3+\ldots)^3 = 3ax(a'x^2+b'x^3+\ldots)$$

whence, equating the co-efficients, we have $3aa'=1$, $b'=0$, etc.

$\therefore$ The branch is $y=x^2/3a+\ldots$ Writing the equation of the osculating conic in the form $y=Ax^2+2Hxy+By^2$, and proceeding as in *Ex.* 1, the expansion for the conic becomes—

$$y=Ax^2+2HAx^3+(2H^2A+A^2B)x^4+\ldots$$

Hence, comparing the terms with those of the branch of the curve we have $A=1/3a$, $H=B=0$

i.e., the osculating conic is $y=x^2/3a$ or $3ay=x^2$.

Similarly, for the other branch, the osculating conic is $3ax=y^2$.

Ex. 3. Find the conic of closest contact at the origin of the curve

$$\alpha x+\beta y^2+y^3=0$$

Ex. 4. The equation of an *n-ic* with a tangent of *n-pointic* contact and a superlinear branch may be put in the form

$$z.y^{n-1}=x^n$$

278. Conics with Four-pointic Contact:

The locus of centres of all conics having a four-point contact with a curve at a given point is a straight line through the point.

Taking the given point as origin, the equation of the curve may be written as—

$$y = ax + bx^2 + cx^3 + dx^4 + \ldots$$

The equation of a conic through the origin having the same tangent may be written as—

$$y = ax + Ax^2 + 2Hxy + By^2$$

The expansion for this is found to be—

$$y = ax + (A + 2aH + Ba^2)x^2$$
$$+ 2(H + aB)(A + 2aH + Ba^2)x^3 + \ldots$$

Since the conic has four-pointic contact, the co-efficients of x, x^2, x^3 must be identical in the equations of the curve and the conic.

$\therefore \quad A + 2aH + Ba^2 = b$ and $2(H + aB)(A + 2aH + Ba^2) = c$,

whence, $\quad H + aB = c/2b, \quad A + aH = b - ac/2b.$

But the centre of the conic is given by—

$$2Ax + 2Hy = a = 0, \qquad 2Hx + 2By - 1 = 0$$

whence $x(A + aH) + y(H + aB) = 0$, *i.e.*, $(2b^2 - ac)x + cy = 0$, which is the locus of the centres, and is called the "axis of aberrancy."

279. Transon's Theory of Aberrancy :*

From what has been said in Chap. IX with regard to the approximate forms of a curve in the neighbourhood of a point, it is clear that the form is, in general, defined by the circle of closest contact, *i.e.*, the circle of curvature. But the form may be further defined by means of the osculating conic.

Let the normal at a given point O on the curve meet at P the infinitesimal chord AB drawn parallel to the tangent at O. Then the arcs OA, OB, as also the lines PA, PB, regarded as small magnitudes of the first order, differ by magnitudes of the second order and may, therefore, be regarded as equal, *i.e.*, if N is the middle point of AB, then NP is a small quantity of the second order. Similarly, OP is also of the second order. The angle $NOP = \tan^{-1}\dfrac{NP}{OP}$ is consequently a finite angle, *i.e.*, the line ON is inclined to the normal OP at a finite angle. In the case of a circle, ON and OP coincide, and therefore the deviation from the circular form is measured by this angle, which is called *aberrancy*,† and the line ON is called the " axis of aberrancy."

We are then led to the following definition :

The measure of deviation of a curve at a given point of it from the circular form is a finite angle, and is called " aberrancy," and the line which forms this angle with the normal is the axis of aberrancy.

When this is a conic, in particular, the axis of aberrancy is the diameter through the point and the aberrancy is the inclination of this diameter to the normal.

* Transon—Recherches sur la courbure des lignes et des surfaces, Liouville, t. VI (1841), pp. 191-207.

† Transon uses the term " deviation," but the term " aberrancy " is generally preferred.

If at a given point of a curve a conic is drawn having four-pointic contact with it, it is clearly seen that the curve and the conic have the same axis of aberrancy, and consequently, the centres of all such conics lie on the axis of aberrancy, as has been directly shown in the preceding article.

The point where the axis of aberrancy at a given point of a curve meets the axis of aberrancy at a consecutive point is called the "*centre of aberrancy*"; consequently, the centre of aberrancy of a curve at a given point is the centre C of the conic having five-pointic contact with the curve at the given point O, *i.e.*, the centre of the osculating conic.* The length OC is called the *radius of aberrancy.*

280. Angle of Aberrancy:

We shall conclude the discussion by referring to only one other important fact in this connection, namely, that the angle of aberrancy δ at any point of a curve is given by the formula †

$$\tan \delta = p - \frac{(1+p^2)r}{3q^2}$$

where p, q, r are the first, second and third differential coefficients of y in regard to x.

* The investigations of these properties properly belong to the differential geometry, and there is no sufficient scope for them in the present work. But in view of the interesting nature of such investigations, it has been considered desirable to refer to some most important points. For further details, the student is required to consult Transon's paper—Liouville Journal, VI (1841), pp. 191-207, and the several papers of A. Mukhopadhyay, Journal of the Asiatic Society of Bengal, Vol. 57, Part II (1888), and Vol. 59, Part II (1890).

† For a second proof of the formula, see the paper by A. Mukhopadhyay, *Differential Equation of a "Parabola,"* J.A.S. of Bengal Vol. 57, Part II, p. 319.

Taking the origin at the given point on the curve, the equation of a conic passing through the origin may be written as—

$$y = mx + ax^2 + 2hxy + by^2 \qquad \ldots \quad (1)$$

If y be expanded in powers of x, by Maclaurin's Theorem, we have—

$$y = px + q\frac{x^2}{2!} + r\frac{x^3}{3!} + \ldots$$

where p, q, $r \ldots$ are respectively the values at the origin of

$$\frac{\partial y}{\partial x}, \qquad \frac{\partial^2 y}{\partial x^2}, \qquad \frac{\partial^3 y}{x^3} \ldots$$

Since the conic has a contact of the third order, the values of p, q, r are the same for the conic and for the curve.

Substituting in the equation of the conic, we have

$$px + q\frac{x^2}{2!} + r\frac{x^3}{3!} + \ldots = mx + ax^2 + 2hx\left(px + q\frac{x^2}{2!} + \ldots\right)$$

$$+ b\left(px + q\frac{x^2}{2!} + \ldots\right)^2$$

Equating the co-efficients, we obtain—

$$m = p, \qquad a + 2hp + bp^2 = \frac{q}{2!}, \quad hq + bpq = \frac{r}{3!}$$

whence, $h + bp = \dfrac{r}{6q}$ and $(a + hp) + p(h + bp) = \dfrac{q}{2}$

i.e., $a + hp = \dfrac{q}{2} - \dfrac{pr}{6q}$ and $\dfrac{a + hp}{h + bp} = \left(\dfrac{3q^2}{r} - p\right) \quad \ldots \quad (2)$

Now, the centre of the conic (1) is given by—

$$2ax+2hy+m=0$$

$$2hx+2by-1=0$$

$\therefore$ The line joining the origin to the centre is—

$$x(a+mh)+y(h+bm)=0 \qquad \dots \quad (3)$$

If δ be the angle which it makes with the normal $my+x=0$, *i.e.*, if δ be the *aberrancy*, we have

$$\tan\delta = \frac{\dfrac{a+mh}{h+bm}-\dfrac{1}{m}}{1+\dfrac{a+mh}{m(h+bm)}}$$

$$=p-\frac{(1+p^2)r}{3q^2}.$$

281. Aberrancy Curve:

Definition: The locus of the centres of osculating conics at points of a given curve is called the *aberrancy curve*.

Taking the tangent and normal at any point of a given curve as axes of co-ordinates, the co-ordinates of the centre of aberrancy may be expressed as—

$$X=R\sin\delta, \quad Y=R\cos\delta$$

where R is the radius of aberrancy, and δ is the angle of aberrancy.

But, from the relation $\tan\delta = p - \frac{(1+p^2)r}{3q^2}$, we obtain

$$\sin\ \delta = \frac{3pq^2 - r(1+p^2)}{\sqrt{1+p^2}\{r^2+(rp-3q^2)^2\}^{\frac{1}{2}}}$$

$$\cos\ \delta = \frac{3q^2}{\sqrt{1+p^2}\{r^2+(rp-3q^2)^2\}^{\frac{1}{2}}}$$

$$\therefore\ X = \frac{3q\{3pq^2 - r(1+p^2)\}}{\sqrt{1+p^2}(3qs-5r^2)}$$

$$Y = \frac{9q^3}{\sqrt{1+p^2}(3qs-5r^2)}$$

where s is the fourth differential co-efficient of y with regard to x.

If, however, instead of taking the tangent and normal as axes of co-ordinates, we take the axes such that the axis of x makes an angle θ with the tangent, we have

$$\tan\ \theta = -\frac{\partial y}{\partial x} = -p$$

$$\sin\ \theta = \frac{-p}{\sqrt{1+p^2}}, \qquad \cos\ \theta = \frac{1}{\sqrt{1+p^2}}$$

and the new co-ordinates (α, β) of the centre of aberrancy are given by the two expressions

$$\alpha = X\cos\theta + Y\sin\theta = \frac{-3qr}{3qs-5r^2}$$

$$\beta = -X\sin\theta + Y\cos\theta = \frac{-3q(pr-3q^2)}{3qs-5r^2}$$

Therefore, when the origin is taken anywhere, the co-ordinates of the centre of aberrancy at any given point (x, y) of the curve are given in the most general form by the formulae *

$$\alpha = x - \frac{3qr}{3qs - 5r^2}$$

$$\beta = y - \frac{3q(pr - 3q^2)}{3qs - 5r^2}$$

Ex. 1. Find the aberrancy curve for the cubic

$$y = ax^3 + 3bx^2 + 3cx + d$$

[The aberrancy curve is $y = \mathrm{A}x^3 + 3\mathrm{B}x^2 + 3\mathrm{C}x + \mathrm{D}$,

where $\mathrm{A} = -ka$, $\mathrm{B} = -kb$, $\mathrm{C} = -kc + (1+k)\dfrac{ac - b^2}{a}$

$$\mathrm{D} = -kd + (1+k)\frac{a^2d - b^3}{a^2}, \quad k = \frac{125}{64}$$

i.e., the aberrancy curve is then a cubic of the same class.

* A. Mukhopadhyay calculated these formulae—J.A.S.B., Vol. 57 (1888), Part II, p. 324, and deduced from them a number of very interesting results, which led to the remarkable geometrical interpretation of Monge's differential equation to all conics, namely,

$$9q^2t - 45qrs + 40r^3 = 0$$

which he denoted by $\mathrm{T} = 0$. It will be interesting to recall in this connection the remarks of Dr. Boole with regard to the geometrical interpretation of Monge's equation (Differential Equations, p. 20)—"But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms." Since then two attempts had been made—one by Lt.-Col. Cunningham, R.E., and the other by Prof. Sylvester—to supply a true geometric interpretation to the *Mongian*, but Mukhopadhyay pointed out the futility of both these interpretations, and gave the following true geometrical interpretation of the Mongian ;—"*The radius of curvature of the Aberrancy Curve vanishes at every point of every conic.*"—J.A.S.B., Vol. 58 (1889), Part I.

At the common points of the curves, we have $(ax+b)^3=0$, which shows that the two curves have only one common point of intersection which is a point of inflexion on both.—A. Mukhopadhyay, Journal of the Asiatic Society of Bengal, Vol. 59, Part II (1890), pp. 61-63.]

Ex. 2. If ρ and ρ' are the radii of curvature of a conic and its evolute, show that the aberrancy in the conic is given by $\tan\phi=\frac{1}{3}\rho'/\rho$ (*cf.* Mukhopadhyay, J.A.S.B., Vol. 57, Part II (1888), pp. 317-319).

Ex. 3. The envelope of the axes of all conics having a four-pointic contact with a curve at a given point is a parabola having the axis of aberrancy for directrix.

Ex. 4. Find the co-ordinates of the centre and the radius of aberrancy for the cubic $y=x^3$ at the point $(1, 1)$.

[The centre of aberrancy is the point $(\frac{8}{5}, -8)$, and the radius is $81\frac{9}{25}$, *i.e.*, the length joining the centre of aberrancy to the point $(1, 1)$.]

Ex. 5. Prove that the aberrancy curve for the curve $y^3=x^2$ is $32y^3=5x^2$ (Math. Tripos, 1891).

Ex. 6. Show that the centre of aberrancy of the curve $y=x^n$ at the point (x, y) is—

$$\left(2\frac{n+1}{2n-1}x,\quad -2\frac{n+1}{n-2}y\right)$$

CHAPTER XIV

SYSTEMS OF CURVES

282. A Pencil of *n-ics*.

Let ϕ and ψ be two curves of order n. The equation $\phi + \lambda\psi = 0$ represents a singly infinite system of curves of order n for different values of λ. Through any point there passes one and only one curve of the system. If one condition is imposed upon the system, it will represent only a finite number of such curves. Thus, the curves of the system having a double point satisfy the equations—

$$\phi_1 + \lambda\psi_1 = 0, \quad \phi_2 + \lambda\psi_2 = 0, \quad \phi_3 + \lambda\psi_3 = 0 \quad \dots \quad (1)$$

where the suffixes indicate differentiation with regard to x, y and z respectively.

If x, y, z be eliminated from the equations (1), the eliminant is of order $3(n-1)^2$ in λ, and gives as many values, corresponding to which there are $3(n-1)^2$ curves in the pencil possessing double points. This number will be reduced, if ϕ and ψ touch.*

But $3(n-1)^2 = n^2 + 4p - 1$. Hence, in a pencil of *n-ics* passing through n^2 base-points, the number of curves with double points is $n^2 + 4p - 1$, where p is the deficiency, and in general, for any pencil with σ base-points, the number is $\sigma + 4p - 1$.†

* Cremona—Einleitung in die Theorie ebener Kurven.

† Cremona—Ann. di mat., Vol. 6(1), (1864), p. 153, and Guccia—Rend. Cir. Math., Vol. 9(1), (1894).

If, however, λ be eliminated between the same equations, we obtain the locus of double points on the system, and these double points are found to have the same polar lines *w.r.t.* all curves of the pencil. Thus we obtain the following theorem :—

In a pencil of n-ics, there are $3(n-1)^2$ *curves with double points, and these double points have the same polar lines with respect to all curves of the system.*

Ex. 1. In a pencil of conics there are three line-pairs, and the pairs intersect in the vertices of a common self-polar triangle.

Ex. 2. In a pencil of cubics through nine points, there are *twelve* curves with double points. The double points have the same polar lines *w.r.t.* all curves of the pencil, and are called the *critic centres* of the system of cubics.

Ex. 3. The locus of inflexions of a pencil of *n-ics* is a $6(n-1)$*-ic*.
[For the eliminant of $\phi+\lambda\psi$ and its Hessian is of order $6(n-1)$.]

Ex. 4. The *k-th* polar curves of any point *w.r.t.* a pencil of *n-ics* form a pencil of $(n-k)$*-ics*.

Ex. 5. The locus of the poles of a given line *w.r.t.* all curves of a pencil of *n-ics* is a $2(n-1)$*-ic*, passing through the points of contact of those curves of the pencil which touch the line.

283. Consider the two curves ϕ and ψ of orders m and n respectively.

The polar line of any point (x', y', z') with respect to the two curves are respectively

$$\left.\begin{array}{l} x\phi_1'+y\phi_2'+z\phi_3'=0 \\ x\psi_1'+y\psi_2'+z\psi_3'=0 \end{array}\right\} \qquad \ldots (1)$$

If these represent the same line, we must have—

$$\phi_1' : \phi_2' : \phi_3' = \psi_1' : \psi_2' : \psi_3'$$

i.e.,

$$\left.\begin{array}{l} \phi_1'\psi_2'-\phi_2'\psi_1'=0 \\ \phi_2'\psi_3'-\phi_3'\psi_2'=0 \\ \phi_3'\psi_1'-\phi_1'\psi_3'=0 \end{array}\right\} \qquad \ldots (2)$$

The common roots of these equations will give the points which have the same polar lines with respect to the curves ϕ and ψ. The first two equations have $(m+n-2)^2$ common roots, but they do not all satisfy the third equation. Again, the $(m-1)(n-1)$ common roots of $\phi_2'=0$ and $\psi_2'=0$ satisfy the first two equations but *not* the third. Hence these roots are to be rejected and the remaining $(m+n-2)^2-(m-1)(n-1)$ roots satisfy all the three equations, *i.e.*, the three equations (2) have—

$$(m+n-2)^2-(m-1)(n-1),$$

or, $$(m-1)^2+(m-1)(n-1)+(n-1)^2$$

common roots.*

Hence, we obtain the theorem :

There are $(m-1)^2+(m-1)(n-1)+(n-1)^2$ points in a plane which have the same polar line with respect to two curves of orders m and n respectively.

Ex. 1. The envelope of the asymptotes of the pencil of *n-ics* $S+kS'=0$ is of class $2n-1$.

Ex. 2. If two curves S and S′ have an *r-ple* point, that point is also an *r-ple* point on the curve $\phi S \pm \psi S'=0$, where ϕ and ψ are any two curves.

284. Curves which touch a Given Curve :

We shall now investigate the number of curves of the pencil $\phi+\lambda\psi=0$ of *n-ics* which touch a given *m-ic* $f=0$.

If (x', y', z') be a point of contact, the tangent to the curve must be the same as the tangent to the curve of the pencil. Hence, we must have—

$$f_1' : f_2' : f_3' = \phi_1'+\lambda\psi_1' : \phi_2'+\lambda\psi_2' : \phi_3'+\lambda\psi_3'$$

whence $$\phi_i'+\lambda\psi_i'+\mu f_i'=0 \quad (i=1, 2, 3) \qquad \ldots \quad (1)$$

where μ is any indeterminate multiplier.

* Salmon—Higher Algebra, § 257.

Eliminating λ, μ, we obtain the locus of (x', y', z') in the form of a determinant equation $J=0$, of order $2n+m-3$.

Hence, the points of contact (x', y', z') are the intersections of the curve $J=0$ with $f=0$, and consequently, their number is $m(2n+m-3)$. Thus, in a pencil of *n-ics* there are $m(2n+m-3)$ curves which will touch a given curve of order m.

If, however, f has δ nodes and κ cusps, **J** behaves as the first polar of f at those points, and the number of intersections is reduced by $2\delta+3\kappa$, *i.e.*, $m(2n+m-3)-2\delta-3\kappa$ curves of a pencil of *n-ics* touch a given *m-ic* with δ nodes and κ cusps. If p be the deficiency of f, the number becomes

$$2(mn+p-1)-\kappa.$$

285. Particular Cases:

Putting $m=1$, we see that $2(n-1)$ curves of the pencil of *n-ics* touch a line. Again, if $n=2$, we obtain two conics of the pencil touching a line.

Putting $n=1$, it follows that there are $m(m-1)$ lines in a pencil which touch a given non-singular *m-ic*, *i.e.*, the class of a non-singular *m-ic* is $m(m-1)$, and that of an *m-ic* with δ nodes and κ cusps is—

$$m(m-1)-2\delta-3\kappa. \quad (\S\ 121.)$$

Putting $n=2$, we obtain $m(m+1)$ conics of a pencil which touch a given *m-ic*. When again $m=2$, six conics of a pencil touch a given conic.

Ex. 1. Deduce Ex. 5, § 282, from the present Article.

Ex. 2. Find the number of circles passing through two given points and touching a given curve.

286. Tact-Invariant of Two Curves :

If we eliminate x', y', z' and μ between the equations (1) of § 284, $\phi' + \lambda\psi' = 0$ and $f' = 0$, we obtain the condition that the pencil $\phi + \lambda\psi$ should touch $f = 0$ in the form $\Theta(\lambda) = 0$, which contains λ in the $m(2n + m - 3)$th degree, and gives the number of curves of the pencil touching f, the value of λ giving the parameters of the curves. The co-efficients of ϕ and ψ each occur in the same degree $m(2n + m - 3)$ and those of f in the degree $n(2m + n - 3)$ in the equation $\Theta(\lambda) = 0$. Hence, we may consider only the co-efficients of ϕ, and obtain the theorem :

The tact-invariant of two curves ϕ and f, of orders n and m respectively is of degree $m(2n + m - 3)$ in the co-efficients of ϕ and of degree $n(2m + n - 3)$ in those of f.

Definition : The condition that two curves should touch is called their *tact-invariant*.

287. Generation of a Curve :

The method of generating conics by means of homographic pencils of lines has been generalised and applied to the case of general *n-ics*, and has actually been applied in generating curves of lower orders.*

Let $\quad u + \lambda v = 0 \quad$ and $\quad \phi + \mu\psi = 0 \quad \ldots \quad (1)$

be two pencils of curves of orders m and n respectively. If the two pencils are so related that to one curve of one system corresponds one, and only one, curve of the other, and *vice versa*, then the parameters are connected by the relation—

$$A\lambda\mu + B\lambda + C\mu + D = 0 \qquad \ldots \quad (2)$$

* Chasles has studied the case of curves of the third order—*Comp. Rendus*, t. 41 (1853). The general case was studied by De Jonquières—Essai sur la generation des courbes géométrique (Memoires présentés par divers savants à l'Académie des sciences.), t. 16 (1858).

The locus of the intersections of corresponding curves is then a curve of order $(m+n)$, whose equation is obtained by eliminating λ and μ between (1) and (2). If $\lambda=\mu$, the locus is represented by the equation $u\psi=v\phi$, which evidently passes through all the base-points of the two pencils.

If P and Q be two base-points of the two pencils respectively, the tangents at these points form two homographic pencils of rays, and there is a projective relation between the pencils of tangents and the pencils of curves.

The locus obtained is the most general curve of order $(m+n)$, and the question whether all algebraic curves can be generated in this manner has been solved by De Jonquières. In fact, for all curves two sets of points may always be determined which may be used as the base-points of two projective pencils for the generation of the curve.

288. The Jacobian of Three Curves:

Consider the three curves $u=0$, $v=0$, and $w=0$ of orders l, m, n respectively.

Then the curve represented by—

$$\mathbf{J}\equiv\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}=0 \qquad \ldots \quad (1)$$

i.e., $\mathbf{J}\equiv\dfrac{\partial\,(u,\ v,\ w)}{\partial\,(x,\ y,\ z)}=0$ is called the *Jacobian* of the three curves u, v, w.

The curve **J** is evidently of order $l+m+n-3$, and since $u_1=u_2=u_3=0$, etc., satisfy the equation, the Jacobian **J** passes through the double points on the three curves.

Again, the Jacobian is the locus of poles whose polar lines with respect to the three curves meet in one point, or the locus of points in which the three first polars of a point intersect.

If three curves u, v, w have a common point, the Jacobian passes through the same point. For, in the determinant (1), multiplying the first column by x, the second by y and the third by z, and adding, we may express this, by Euler's theorem, as—

$$\mathbf{J}x \equiv \begin{vmatrix} lu & u_2 & u_3 \\ mv & v_2 & v_3 \\ nw & w_2 & w_3 \end{vmatrix} = 0 \qquad \dots \quad (2)$$

which proves the property.

In general, if a point is a *q-ple* point on u, *r-ple* point on v and *s-ple* point on w, that point is a multiple point of order not less than $(q+r+s-2)$ on the Jacobian.*

Again, the Jacobian passes through the points of contact of the pencil $u \pm \lambda v = 0$, and the curve $w=0$, as has already been shown in the particular case, when $l=m$ (§ 284).

Writing the determinant (2) in the form—

$$\mathbf{J}.x \equiv lu.\phi_1 + mv.\phi_2 + nw.\phi_3,$$

where ϕ_1, ϕ_2, ϕ_3 are the corresponding co-factors, and differentiating *w.r.t.* x, we obtain—

$$x\frac{\partial \mathbf{J}}{\partial x} + \mathbf{J} = lu_1\phi_1 + mv_1\phi_2 + nw_1\phi_3$$
$$+ lu\frac{\partial \phi_1}{\partial x} + mv\frac{\partial \phi_2}{\partial x} + nw\frac{\partial \phi_3}{\partial x}$$

* Cremona—Introduzioni, etc., § 93. Also Guccia—Rendiconti Circolo Mat. di Palermo, Vol. 7 (1893), p. 193.

$\therefore$ When $u=v=w=\mathrm{J}=0$, and $l=m$, we have—

$$x\frac{\partial \mathrm{J}}{\partial x}=l(u_1\phi_1+v_1\phi_2+w_1\phi_3)+(n-l)w_1\phi_3$$

$$=l\mathrm{J}+(n-l)\ w_1\phi_3=(n-l)w_1\phi_3.$$

Similarly, $x\dfrac{\partial \mathrm{J}}{\partial y}=(n-l)w_2\phi_3$, and $x\dfrac{\partial \mathrm{J}}{\partial z}=(n-l)w_3\phi_3$

whence, $\left(x'\dfrac{\partial}{\partial x}+y'\dfrac{\partial}{\partial y}+z'\dfrac{\partial}{\partial z}\right)\mathrm{J}$

$$=\frac{(n-l)\phi_3}{x}(x'w_1+y'w_2+z'w_3)$$

which shows that **J** and w have the same tangent at the point (x, y, z).

It can be further proved that the Jacobian passes through the node on w, and has a node at a cusp on w.

289. Net of Curves:

If u, v, w are any three curves of order n, the doubly infinite system of curves $\lambda u+\mu v+\nu w=0$ is called a "net" of curves of order n. Certain properties of the net are at once evident. Any curve of the net is uniquely determined by two points, and consequently, all curves through any point form a pencil; for, substituting the co-ordinates of the point in the equation of the net, we may eliminate one of the parameters μ/λ, v/λ. Hence, the system of *n-ics* through $\frac{1}{2}n(n+3)-2$ fixed points forms a net, and the equation of an *n-ic* passing through $\frac{1}{2}n(n+3)-2$ points can be put into the above form.

290. The Jacobian of a Net of Curves:

The co-ordinates of the double points of any curve of the net $\lambda u+\mu v+\nu w=0$ satisfy the three equations—

$$\lambda u_1+\mu v_1+\nu w_1=0$$

$$\lambda u_2+\mu v_2+\nu w_2=0$$

$$\lambda u_3+\mu v_3+\nu w_3=0$$

Eliminating λ, μ, ν between them, the locus of the double points of the system is the Jacobian—

$$\mathbf{J}\equiv\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}=0$$

which is a curve of order $3(n-1)$. Hence we obtain the theorem:

The locus of double points of a net of curves of order n, passing through $\frac{1}{2}n(n+3)-2$ fixed points is a curve of order $3(n-1)$.

Since a common point of u, v, w is a double point on the Jacobian, the $\frac{1}{2}n(n+3)-2$ fixed base-points of the net are double points on the Jacobian.

Ex. 1. The nodes of all nodal cubics through seven given points lie on a sextic curve having those seven points as double points.

Ex. 2. If the curves of a net have an *r-ple* point, that point is a $(3r-1)$-*ple* point on the Jacobian.

Ex. 3. Show that the Jacobian is the locus of the nodes of the family $\lambda vw+\mu wu+\nu uv=0$.

291. Net of First Polars:

The first polar of any point (x', y', z') with regard to any curve $\phi=0$ is

$$x'\phi_1+y'\phi_2+z'\phi_3=0$$

which is a curve of order $n-1$. If now x', y', z' are regarded as parameters, this represents a *net*, whose base-curves are

$$\phi_1=0, \quad \phi_2=0, \quad \phi_3=0.$$

The Jacobian of this net is, therefore, the Hessian of the original curve $\phi=0$. Thus the *Hessian of a curve is the Jacobian of the net of first polars.*

Each *r-ple* point on ϕ is an $(r-1)$*-ple* point on the first polars, and this again is a multiple point of order $3(r-1)-1$, *i.e.*, $3r-4$ on the Jacobian, *i.e.*, the Hessian of ϕ (§ 105). Since r tangents of the Hessian coincide with those of the original curve, the point counts as $r(3r-4)+r=3r(r-1)$ intersections. Hence, at each *r-ple* point coincide $3r(r-1)$ inflexions of the curve, and it is equivalent to $\frac{1}{2}r(r-1)$ double points.

The locus of points, whose first polars with regard to the curves of the net of order n have a common point, is a curve S of order $3(n-1)^2$, and is called the *Steinerian* of the net.

The Jacobian and the Steinerian have a (1, 1) correspondence, and the lines joining corresponding points on the two curves envelop a curve of class $3n(n-1)$, which is called the *Cayleyan* of the net.*

More generally, *if two curves of orders n and n' have a one-to-one relation, the lines joining the corresponding points envelop a curve of class $n+n'$.*

* For covariant curves of a net, see E. Kötter—Math. Ann. Bd. 34 (1889), p. 123, and Scott—Quarterly Journal, Vol. 29 (1898), p. 329, and Vol. 32 (1900), p. 209.

This theorem can be easily proved by means of Chasles' correspondence principles after the method of § 250.

Ex. 1. Show that the Jacobian of a net of circles is a circle belonging to the net of orthogonal circles.

Ex. 2. The Jacobian of a net of conics, having a common self-polar triangle, reduces to the three sides of the triangle.

Ex. 3. If three conics touch at a common point, the Jacobian reduces to the common tangent and a conic. If they have a three-pointic contact, the Jacobian is the common tangent taken thrice, while in the case of a four-pointic contact, the Jacobian vanishes identically.

292. Invariants and Covariants of two Ternary Forms:

Consider the *n-ic* $\phi(x, y, z)=0$ ... (1)

and the line $lx+my+nz=0$ ... (2)

The polar conic of a point $P(x', y', z')$ with respect to the *n-ic* is $\left(x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}\right)^2 \phi=0$

i.e., $S' \equiv a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0$ ... (3)

where a' b', c',....represent the second differential co-efficients *w. r. t.* x', y', z'.

The condition that (2) touches (3) gives an equation of order $2(n-2)$ in (x', y', z'), whence we obtain :—

The locus of points whose polar conics touch a given line is a $2(n-2)$-ic, which is again enveloped by the polar conic of any point on the line.

Similarly, the condition that the polar cubic of P touches the line gives an equation of order $4(n-3)$, and consequently, the locus of P is a $4(n-3)$-*ic*.

Again, if $\mathbf{S} \equiv ax^2 + by^2 + cz^2 + \ldots \qquad = 0 \quad \ldots \quad (4)$

be a given conic, there are two mixed invariants * of (3) and (4), namely, Θ and Θ', of orders $(n-2)$ and $(2n-4)$ respectively in (x', y', z').

But $\Theta' = 0$ expresses the fact that a triangle inscribed in $\mathbf{S}$ is self-conjugate *w.r.t.* the polar conic $\mathbf{S}'$,* or, that the triangle circumscribed to $\mathbf{S}'$ is self-polar *w.r.t.* $\mathbf{S}$.

Hence, *the locus of a point, whose polar conic is inscribed in a triangle self-polar w.r.t. to a given conic, or whose polar conic has a self-polar triangle inscribed in a given conic is a* $2(n-2)$*-ic.*

Similarly, $\Theta = 0$ expresses the fact that an inscribed triangle of $\mathbf{S}'$ is self-polar *w.r.t.* $\mathbf{S}$, or the triangle circumscribed about $\mathbf{S}'$ is self-polar *w.r.t.* to $\mathbf{S}$.

Thus, *the locus of a point, whose polar conic is inscribed in or circumscribed about a triangle self-polar for a given conic is an* $(n-1)$*-ic.*

The condition of contact of (3) and (4) gives an equation of order $6(n-2)$ in (x', y', z'), whence *the locus of points whose polar conics touch a given conic is a curve of order* $6(n-2)$.

Again, if $\mathbf{S}'$ breaks up into two right lines, $\Theta' = 0$ expresses the fact that the intersection of the lines lies on $\mathbf{S}$,† while the locus of (x', y', z') is the Hessian. Thus the locus of points whose polar conics break up into two lines intersecting on a given conic is a $2(n-2)$*-ic*.

Ex. 1. Shew that there are $6(n-2)^2$ points on the Hessian whose polar conics are right lines intersecting on a given conic.

Ex. 2. There are $3(n-2)^2$ points on the Hessian whose polar conics are conjugate lines for a given conic.

Ex. 3. There are $6(n-2)^3$ points on the Hessian whose polar conics touch a given conic.

* Salmon—Conics, Chap. XVIII, p. 334.

† Salmon—Conics, § 375, p. 340.

293. Characteristics of a System of Curves:

A singly infinite system of curves may be algebraically represented by means of an algebraic equation whose co-efficients are functions, not necessarily rational, of a parameter, or if irrational, they may be expressed rationally in terms of two parameters connected by an algebraic equation, as is shown in the theory of functions. De Jonquières * considered the properties of a system of curves of order n satisfying $\frac{1}{2}n(n+3)-1$ conditions, *i.e.*, one less than the number sufficient to determine an n-*ic*, and the family of curves thus represented is characterised by the number of curves which pass through an arbitrary point, and if the parameter enters the equation in degree μ, μ † gives the number of such curves of the family and is called its *characteristic*. Chasles,‡ however, uses *two* characteristics, namely, the number μ of curves of the family which pass through an arbitrary point, and the dual number ν of curves which touch an arbitrary line, which in fact, is the degree in which the parameter enters the line-equation ϕ of the system. Since ϕ is of degree $2(n-1)$ in the co-efficients of the point-equation f, we have $\nu=2\mu(n-1)$.

Cayley calls μ and ν the *parametric order* and *class* respectively of the family.

Ex. 1. Find the characteristics of a pencil of n-*ics*. Since only one curve passes through any point, $\mu=1$ and $\nu=2\mu(n-1)=2n-2$.

Ex. 2. What are the characteristics of a family of conics touching two given lines at given points? [1, 1]

Ex. 3 Shew that the characteristics of the polar reciprocal of a family of curves whose characteristics are (μ, ν) are (ν, μ).

* De Jonquières—Liouville Journal, t. 6(2), (1861), p. 113.

† Cayley has shown that the converse is not always true—Phil. Trans. Lond., Vol. 158 (1868), or Coll. Works, Vol. 6, p. 191.

‡ Chasles—Papers in Comp. Rend., Vols. 58 and 59 (1864-67). The theory of characteristics is due to Chasles, and was afterwards developed by Jonquières, Cayley, Salmon, Zeuthen, etc.

294. Relation between the Characteristics :

Let P, Q,...be the points in which a curve of the family (μ, ν) of order n meets a given line. Since μ curves of the family pass through P, each meeting the line in $n-1$ other points, to each point P correspond $\mu(n-1)$ points Q, and similarly, to each point Q correspond $\mu(n-1)$ points P. There is then a $\{\mu(n-1), \mu(n-1)\}$ correspondence on the line, and the united points of the correspondence, $2\mu(n-1)$ in number, are the points of contact of the curves of the family with the line,

i.e., $$\nu=2\mu(n-1).$$

But a curve of the series may be a complex containing a portion counted twice. Hence, for proper contact the number of united points arising from such curves must then be deducted from $2\mu(n-1)$. In the case of conics, if the number of coincident right lines in the system be λ,

since $n=2$, $\quad \nu=2\mu(2-1)-\lambda=2\mu-\lambda.$

Reciprocally, if ω be the number of point-pairs in a range of conics, $$\mu=2\nu-\omega.$$

In particular, a system of conics satisfying four conditions contains $2\nu-\mu$ line-pairs and $2\mu-\nu$ point-pairs, and Zeuthen's * investigations are based upon these facts, and he takes λ, ω as the characteristics of the system of conics instead of μ, ν, it being easier, in most cases, to ascertain the number of conics of a given system which reduce to line-pairs or point-pairs, than the number which pass through any arbitrary point or touch any given line.

Thus $\quad \mu=\frac{1}{3}(2\lambda+\omega)$ and $\nu=\frac{1}{3}(2\omega+\lambda)$

* Zeuthen—Comp. Rend., Vol. 89 (1879), p. 899, etc.

The relation between Chasles' numbers μ, ν and Zeuthen's numbers λ, ω can be clearly seen by considering the conditions satisfied by a system of conics through four given points, or through three given points and touching a line, and so on.

The values of μ, ν, λ, ω, corresponding to the different cases of a system of conics, are found as in the scheme:

	μ	ν	λ	ω
(: :)	1	2	0	3
(. :/)	2	4	0	6
(://)	4	4	4	4
(·///)	4	2	6	0
(////)	2	1	3	0

For further information and details with regard to characteristics of curves satisfying given conditions, the reader is referred to the papers of Cayley above referred to, and to Clebsch—Leçons sur la Geometrie, Vol. II, pp. 113-129, Brill—Math. Ann. Bd. 10 (1876), p. 534, and Halphen—Liouville Journal, Vol. 2 (3), (1876).

Ex. 1. The locus of the poles of a given line *w.r.t.* the family (μ, ν) is a ν-*ic*, and the envelope of the polars of a given point *w.r.t.* the curves of the system is a curve of class μ.

Ex. 2. Shew that the locus of a point whose polar *w.r.t.* a fixed curve of order n' and class m' coincides with its polar with respect to some curve of a family (μ, ν) is a curve of order $\nu + \mu(m'-1)$.

To determine the order of the curve, we consider its intersection with a line. Consider two points P and Q on the line such that the

polar of P *w.r.t.* the fixed curve coincides with that of Q *w.r.t.* some curve of the family.

If P and Q coincide, we have the condition of the problem satisfied. Suppose P is fixed. Then the locus of the poles of its polar *w.r.t.* the curves of the family is of order ν, and hence corresponding to any position of P there are ν positions of Q. Again, suppose Q is fixed. Then its polars *w.r.t.* the curves of the family envelop a curve of class μ; and since the polars of points on the line *w.r.t.* the given curve envelop a curve of class $n'-1$, there are $\mu(n'-1)$ common tangents to the two envelopes and each corresponds to a position of P. Thus, there is a $\{\nu, \mu(n'-1)\}$ correspondence on the line, and there are $\nu+\mu(n'-1)$ united points. Hence the order of the required locus is $\nu+\mu(n'-1)$.

Ex. 3. Find the number of curves of a family (μ, ν) which touch a given curve of order n' and class m'.

[The locus in Ex. 2 meets the fixed curve in $n'\{\nu+\mu(n'-1)\}$ or $n'\nu+m'\mu$ points, each of which is a point of contact of the fixed curve with a curve of the family. Hence the required number is $m'\mu+n'\nu$.]

Ex. 4. Shew that the locus of the points of contact of two curves of the families (μ_1, ν_1), (μ_2, ν_2) is a curve of order

$$\mu_1\nu_2+\mu_2\nu_1+\mu_1\mu_2.$$

Ex. 5. Find the characteristics of cubics with a given cusp, inflexion, tangent and its point of contact.

[The equation of the curves of the family may be written as

$$z(y+ax)^2=axy^2+2a^2x^2y.$$

Hence, the characteristics are (2, 3).]

Ex. 6. Shew that the characteristics of cubics with nine given inflexions are (1, 4).

Ex. 7. The characteristics of quartics with three given nodes, and drawn through four given points are (1, 6).

295. The Characteristics of Conditions:

The number of curves of a system which satisfy any other condition will, in general, be of the form $\mu\alpha+\nu\beta$, where α, β are independent of μ, ν and are called the *characteristics* of the condition.

This was given by Chasles in the case of conics, but the general theorem was proved by Clebsch * and Halphen † and applied to higher curves. If a curve be determined by a sufficient number of conditions of any kind, and the characteristics for each condition be given, we can determine the number of curves satisfying the prescribed condition.

Consider the case of a system of conics. The number of conics determined by five given points, by four points and a tangent, by three points and two tangents, etc., is determined symbolically as follows :

(:·:), (: :/), (∴//), (:///), (·////), (/////)

1 2 4 4 2 1

Consequently, the characteristics of the systems determined by four points, three points and a tangent, etc., are—

(: :), (∴/), (://), (·///), (////)

1, 2 2, 4 4, 4 4, 2 2, 1

The number of conics satisfying the conditions whose characteristics are α, β and also passing through four points, three points and touching a line, etc., are—

$$\alpha+2\beta,\ 2\alpha+4\beta,\ 4\alpha+4\beta,\ 4\alpha+2\beta,\ 2\alpha+\beta$$

These numbers, in fact, are not independent, but are connected by three relations.‡

* Clebsch—Math. Ann., Vol. 6 (1873), p. 1.

† Halphen—Bull. Soc. Math. de France, Vol. I (1873), pp. 130-141, and Proc. Lond. Math. Soc., Vol. 9 (1878).

‡ Salmon—H. P. Curves, § 413.

We may, however, establish the following more general theorem :

In a system (μ, ν) *of curves, there are* $n'\nu + m'\mu$ *curves which touch a given curve* C_n' *of order* n' *and class* m'.

Suppose the given curve C_n' consists of a pencil of n' right lines passing through any point P, which is, therefore, to be regarded as an m'-*ple* point.

Now, the required curves of the system are :—

(1) those which touch any of the n' lines, giving $n'\nu$ curves,

(2) all curves through P, each being counted m' times, since P is to be regarded as an m'-*ple* point. This gives $m'\mu$ curves.

Hence, the total number of curves of the family touching an n'-*ic* of class m' is $n'\nu + m'\mu$, as was otherwise found in Ex. 3, § 294.

For a detailed account of the theory, the student is referred to the original papers quoted above, and to Cayley's Paper—" On the curves which satisfy given conditions "— Coll. Works, Vol. 6, pp. 200-207.

Ex. 1. Find the number of conics in the above five cases, which touch a given conic, which is of order 2 and class 2.

From § 285, we have $\alpha + 2\beta = 6$, and by the principle of duality, $2\alpha + \beta = 6$, whence $\alpha = 2 = \beta$, and the number of conics in the five different cases are 6, 12, 16, 12, 6.

Ex. 2. The locus of the points of contact of tangents drawn from a fixed point to a system (μ, ν) is a curve of order $\mu + \nu$, having a μ-*ple* point at the fixed point.

Ex. 3. Deduce the results of § 291 from the formula $n\nu + m\mu$, by putting $\mu = 1$, $\nu = 2(n-1)$. In the case of a conic the number becomes $2(\mu + \nu)$.

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